5 Integration

5.1 Antiderivatives

∘ A function F is called an *antiderivative* of f if $\frac{d}{dx}F(x) = f(x)$. (F is also known as an *indefinite integral* of f and denoted by $\int f(x)\ dx$.)

∘ If F is an antiderivative of f then the most general antiderivative of f if $F(x) + c$ where c is an arbitrary constant.

∘ Common antiderivatives

$f(x)$	$F(x)$
$cf(x)$	$cF(x)\ (F' = f)$
$f(x) + g(x)$	$F(x) + G(x)\ (F' = f, G' = g)$
x^n	$\frac{x^{n+1}}{n+1} + c,\ (n \neq -1)$
$\frac{1}{x}$	$\ln \lvert x \rvert + c$
e^x	$e^x + c$

$f(x)$	$F(x)$
$\sin x$	$-\cos x$ +c
$\cos x$	$\sin x + c$
$\sec^2 x$	$\tan x + c$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$
$\frac{1}{1+x^2}$	$\tan^{-1} x + c$

Exercise 1. Find the most general antiderivative of each of the following functions. (Check your answer by differentiation.)

1. $f(x) = 2$ 2. $f(x) = 3x + 5$ 3. $f(x) = x^2 - x^{-2}$ 4. $f(x) = 3x^{3/5} + 4x^{-2/5}$

Exercise 2. Find the most general antiderivative of each of the following functions. (Check your answer by differentiation.)

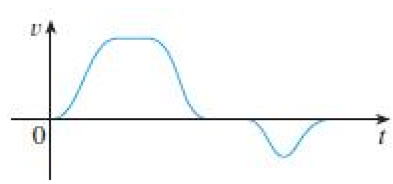
1. $f(x) = \cos 2x$ 2. $f(x) = 2e^{3x}$ 3. $f(x) = \sec^2 3x - (1-x^2)^{-1}$ 4. $f(x) = 2x^{-1} - 3\sin 4x$

Exercise 3. Find f: 1. $f''(x) = x^3 - 2x^2 + 5$ 2. $f'''(x) = \cos x$

Exercise 4. Find f: 1. $f'(x) = 1 + 3\sqrt{x}$, $f(4) = 25$ 2. $f''(\theta) = \sin\theta + \cos\theta$, $f(0) = 3, f'(0) = 4$

Exercise 5. What constant acceleration is required to increase the speed of a car from 30 km/h to 50 km/h in 5 s?

Exercise 6. The graph of the velocity function of a particle is shown in the figure. Sketch the graph of a position function.



5.2 Definite Integral

Definition:
Let f be a continuous function defined on $a \leq x \leq b$ and let $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be endpoints of n sub-intervals of interval $[a, b]$. Let $\Delta x = \frac{b-a}{n}$ and $x_i^\star \in [x_{i-1}, i]$ $(i = 1, 2, \cdots, n)$.

Then, the *definite integral* of f from a to b is

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x$$

provided that this limit exists. It Gives the same value for all possible choices of sample points $x_i^\star$. If it does exist, we say that f is *integrable* on $[a, b]$.

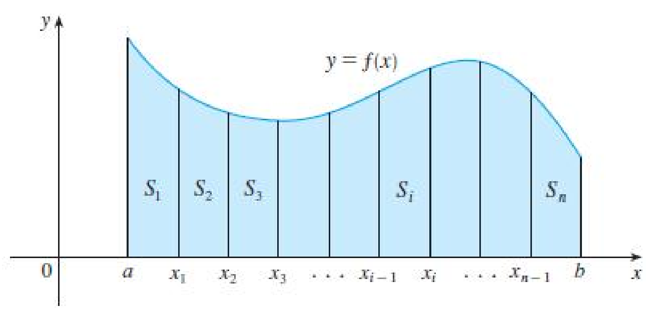
Note: The quantity $\sum_{i=1}^n f(x_i^\star)\Delta x$ is known as *Riemann sum.*

Note: The definite integral $\int_a^b f(x)\ dx$ is a number and it does not depend on x.

Theorem: If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\ dx$ exists.

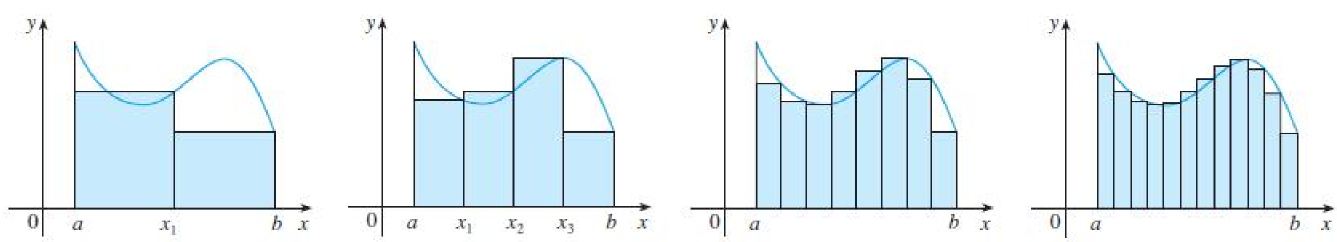
5.3 Definite Integral and Area

Exercise 7. Suppose we are given the function f shown here and we want to find the (shaded) area of the region bounded by the vertical lines $x = a$ and $x = b$, the $x-$axis and the graph of f.
(a) Give an algebraic expression that approximates the shaded area.
(b) Give an algebraic expression that is equal to the shaded area.



Note: We may choose $x_i^\star$ to be the left end point, right end point, mid point or any other point in the interval.

Approximated area for $n = 2, 4, 8, 12$ with $x_i^\star$ chosen to be the right end point of each interval;



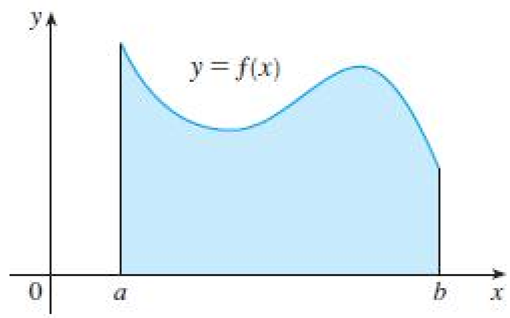
(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

Exercise 8. Suppose the odometer on a car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

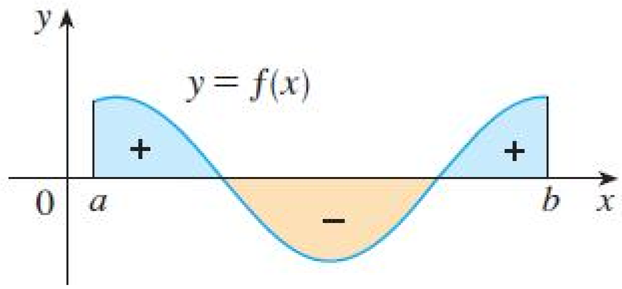
Time (s)	0	5	10	15	20	25	30
Velocity (m/s)	17	21	24	29	32	31	28

(a) Estimate the distance traveled during 30s.
(b) Sketch a graph of velocity vs time and explain the relation among area under the graph, distance traveled, and the definite integral $\int_0^{30} v(t)\, dt$ where $v(t)$ denotes the velocity at time t.

Note: If $f \geq 0$ for all $x \in [a, b]$ then the definite integral $\int_a^b f(x)\, dx$ represents the area under the curve $y = f(x)$, above the $x-$axis, from a to b.

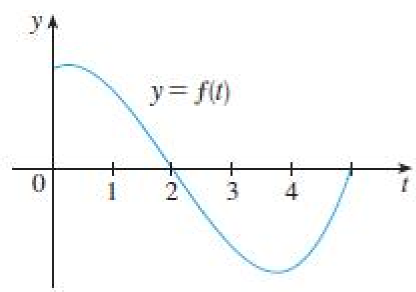


Note: If f takes on both positive and negative values for $x \in [a, b]$ then the definite integral $\int_a^b f(x)\, dx$ represents the ***net area*** under the curve $y = f(x)$, above the $x-$axis, from a to b.



Exercise 9. Evaluate $\int_0^1 \sqrt{1 - x^2}\, dx$ by interpreting in terms of area.

Exercise 10. If $F(x) = \displaystyle\int_0^x f(t)\, dt$ where f is the function whose graph is given, estimate $F(i), i = 0, 1, \cdots 5$.



5.4 Properties of the Definite Integral

1. $\displaystyle\int_a^a f(x)\,dx = 0$

2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\displaystyle\int_a^b c\,dx = \int_a^b c(b-a)\,dx$ where c is an arbitrary constant.

4. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ where c is an arbitrary constant.

5. $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

6. $\displaystyle\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

7. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

8. If $f(x) \geq 0$ for $a < x < b$ then $\displaystyle\int_a^b f(x)\,dx \geq 0$.

9. If $f(x) \geq g(x)$ for $a < x < b$ then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

10. If $m \leq f(x) \leq M$ for $a < x < b$ then $\displaystyle m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Exercise 11. If $\int_1^5 f(x)\,dx = 12$ and $\int_4^5 f(x)\,dx = 5$ find $\int_1^4 f(x)\,dx$.

Exercise 12. Find $\int_0^5 f(x)\,dx$ if $f(x) = \begin{cases} 3 & \text{if } x < 3 \\ x & \text{if } x \geq 3 \end{cases}$.

Exercise 13. Show that $2 \leq \int_{-1}^1 \sqrt{1+x^2}\,dx \leq 2\sqrt{2}$.

5.5 Evaluating Definite Integrals (Approximating by Numerical Methods)

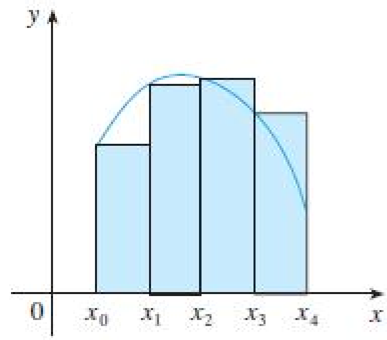
Suppose that we divide the interval $[a, b]$ into n sub-intervals of equal length $\Delta x = \dfrac{b-a}{n}$, and $x_i^\star$ is any point in the ith sub-interval $[x_{i-1}, x_i]$. Then we have

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^\star)\Delta x.$$

1. Left endpoint approximation:

The point $x_i^\star$ is chosen to be the left endpoint of the interval. Then,

$$\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

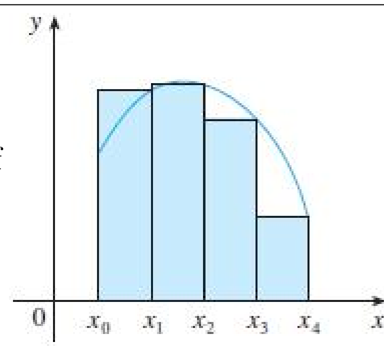


(a) Left endpoint approximation

2. Right endpoint approximation:

The point $x_i^\star$ is chosen to be the right endpoint of the interval. Then,

$$\int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$



(b) Right endpoint approximation

∘ The *error* in using an approximation is defined to be the amount that needs to be added to the approximation to make it exact.

3. Midpoint rule:

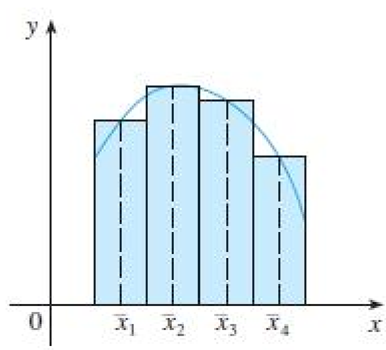
The point $x_i^\star$ is chosen to be the mid point $\overline{x}_i$ of the interval. Then,

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x.$$

Error bound: Error E_M of mid point rule is bounded by

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

where $|f''(x)| \leq K$ for $a \leq x \leq b$.



(c) Midpoint approximation

4. Trapezoidal rule:

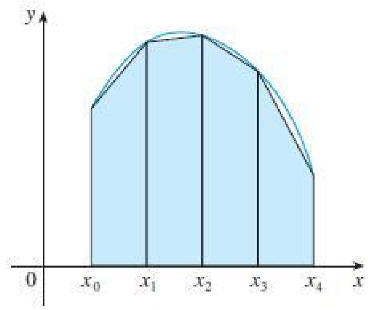
Averaging left and right end approximations,

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)].$$

Error bound: Error E_T of trapezoidal rule is bounded by

$$|E_T| \le \frac{K(b-a)^3}{12n^2}$$

where $|f''(x)| \le K$ for $a \le x \le b$.



Trapezoidal approximation

5. Simpson's rule:

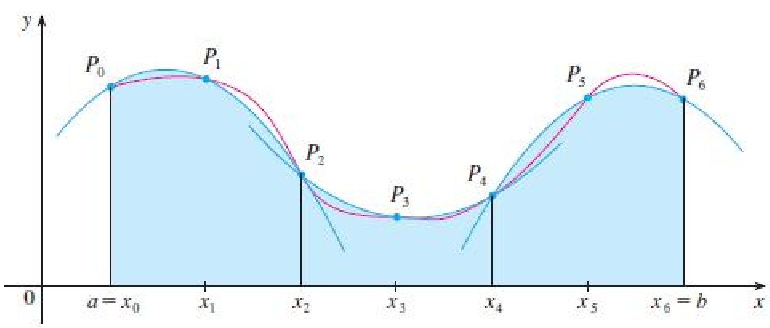
Let n be even. Using parabolas to approximate the curve (instead of lines),

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)].$$

Error bound: Error E_S of Simpson's rule is bounded by

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

where $|f^{(4)}(x)| \le K$ for $a \le x \le b$.



Exercise 14.
(a) Find the exact value of the definite integral $\int_1^2 \frac{1}{x}\,dx$. (Give you answer to 6 decimal places.)

(b) Find the approximated value of the definite integral $\int_1^2 \frac{1}{x}\,dx$ using each method given above. (Give you answer to 6 decimal places.)
(i) Use left end approximation with $n = 5, 10$ and 20.
(ii) Use right end approximation with $n = 5, 10$ and 20.
(iii) Use mid point approximation with $n = 5, 10$ and 20.
(iv) Use trapezoidal rule with $n = 5, 10$ and 20.
(v) Use Simpson's rule with $n = 10$ and 20.

Note: Error = [exact value of $\int_1^2 \frac{1}{x}\,dx$] − [approximated value].

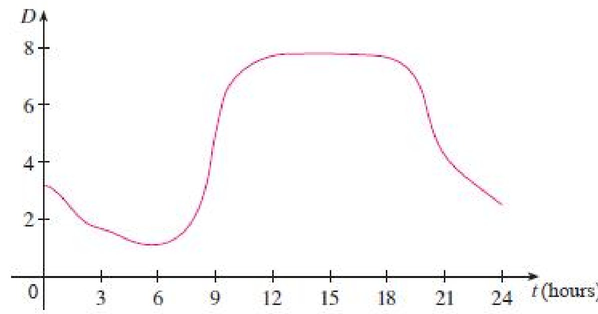
(c) Calculate the error for each calculation in b(i)-b(v).

(d) Calculate error bounds for mid point rule, trapezoidal rule, and Simpson's rule.

(e) Comment on your answers in parts b,c, and d.

Exercise 15. How large should we take n in order to guarantee that the approximations for the definite integral $\int_1^2 \frac{1}{x}\,dx$ are accurate to within 0.0001 if we use; (a) mid point rule (b) trapezoidal rule (c) Simpson's rule?

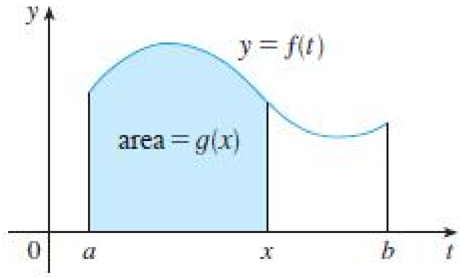
Exercise 16. Figure shows data traffic on the link from the United States to SWITCH, the Swiss academic and research network, on February 10, 1998. $D(t)$ is the data throughput, measured in megabits per second (Mb/s). Estimate the total amount of data transmitted on the link from midnight to noon on that day.



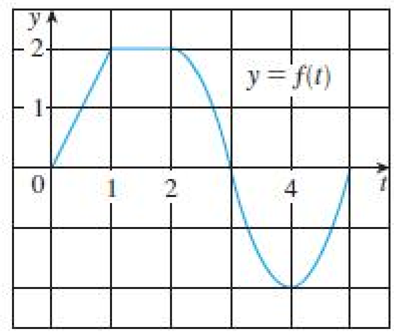
5.6 The Fundamental Theorem of Calculus

∘ Area function ("area so far")

Let $g(x) = \displaystyle\int_a^x f(t)\, dt$ where f is continuous and $a \le x \le b$. The function g is known as the area (so far) function; area under the graph of f from a to x, where x can vary from a to b.



Exercise 17. If f is the function whose graph is shown and $g(x) = \int_0^x f(t)\, dt$ find $g(0), g(1), g(2), g(3), g(4), g(5)$ and sketch a rough graph of g.



The Fundamental Theorem of Calculus (FTC)

Suppose f is continuous on $[a, b]$. Then,

Part 1. If $g(x) = \displaystyle\int_a^x f(t)\, dt$ then $g'(x) = f(x)$. Alternatively $\displaystyle\frac{d}{dx}\int_a^x f(t)\, dt = f(x)$.

Part 2. $\displaystyle\int_a^b f(x)\, dx = F(b) - F(a)$ where F is any antiderivative of f, i.e., $F' = f$.

Alternatively $\displaystyle\int_a^b F'(x)\, dx = F(b) - F(a) = F(x)\Big|_a^b$.

Exercise 18. Apply the FTC1 to find the derivative of each function.

(a) $g(x) = \displaystyle\int_1^x \frac{1}{1+u^3}\, du$ (b) $g(x) = \displaystyle\int_x^0 \sqrt{1+t^2}\, dt$

(c) $g(x) = \displaystyle\int_0^{x^2} \cos^3\theta\, d\theta$ (d) $g(x) = \displaystyle\int_1^{x^4} \sec t\, dt$

Exercise 19. Apply the FTC2 to evaluate each definite integral.

(a) $\displaystyle\int_{-1}^1 x^3\, dx$ (b) $\displaystyle\int_0^4 (4-x)\sqrt{x}\, dx$ (c) $\displaystyle\int_1^2 \frac{1+u^4}{u^2}\, du$ (d) $\displaystyle\int_0^{\pi/4} \sec\theta\tan\theta\, d\theta$

Exercise 20. True or false? $\displaystyle\int_{-1}^3 \frac{1}{x^2} = -\frac{4}{3}$. Justify your answer.

Exercise 21. The sine integral function is defined as $Si(x) = \displaystyle\int_0^x \frac{\sin t}{t}\, dt.$

[The *integrand* $f(t) = \frac{\sin t}{t}$ is not defined when $t = 0$, but we know that its limit is 1 when $t \to 0$. So we define $f(0) = 1$ and this makes f a continuous function everywhere.]

(a) Draw the graph of Si.
(b) At what values of does this function have local maximum values?
(c) Find the coordinates of the first inflection point to the right of the origin.
(d) Does this function have horizontal asymptotes?
(e) Solve the following equation correct to one decimal place: $\int_0^x \frac{\sin t}{t}\, dt = 1$

(Use any computational software.)