6 Techniques of Integration

6.1 Method of substitution

The substitution rule for indefinite integrals:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Exercise 1. Use a suitable substitution to evaluate each indefinite integral.

1. $\int \sqrt{2x+1}\ dx$
2. $\int (x+1)\sqrt{2x+x^2}\ dx$
3. $\int x^3(2+x^4)^5\ dx$
4. $\displaystyle\int \frac{2^t}{2^t+3}\ dt$
5. $\int x^5\sqrt{1+x^2}\ dx$
6. $\int x^3\cos(x^4+2)\ dx$
7. $\int x^2 e^{x^3}\ dx$
8. $\displaystyle\int \frac{(\ln x)^2}{x}\ dx$
9. $\int e^{\cos\theta}\sin\theta\ d\theta$
10. $\displaystyle\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\ dx$
(a and b are constants)

The substitution rule for definite integrals:

If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

Exercise 2. Use a suitable substitution to evaluate each definite integral.

1. $\int_0^4 \sqrt{2x+1}\ dx$
2. $\int_1^2 x\sqrt{x-1}\ dx$
3. $\int_0^{\pi/2} \cos x\sin(\sin x)\ dx$
4. $\int_0^{\sqrt{\pi}} x\cos(x^2)\ dx$
5. $\displaystyle\int_0^1 \frac{1+e^z}{z+e^z}\ dz$
6. $\int_0^{\pi} \sec^2(t/4)\ dt$
7. $\int_0^4 x(1+2x)^{-\frac{1}{2}}\ dx$
8. $\int_{1/2}^1 x^{-3}\cos(x^{-2})\ dx$
9. $\int_0^a x\sqrt{a^2-x^2}\ dx$
10. $\int_{-\pi/3}^{\pi/3} x^4\sin x\ dx$

6.2 Integration by parts

Integration by parts formula:

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int g(x)f'(x)\ dx$$

Exercise 3. Use integration by parts to evaluate each integral.

1. $\int x \sin x\ dx$
2. $\int \ln x\ dx$
3. $\int x^2 e^x\ dx$
4. $\int e^x \sin x\ dx$
5. $\int \arcsin x\ dx$
6. $\int x^2 \ln x\ dx$
7. $\int (\ln x)^2\ dx$
8. $\int e^{2\theta} \sin 3\theta\ d\theta$
9. $\int (\cos x) \ln \sin x\ dx$
10. $\displaystyle\int \frac{xe^{2x}}{(1+2x)^2}\ dx$

Exercise 4. Use integration by parts to evaluate each definite integral.

1. $\int_0^1 \tan^{-1} x\ dx$
2. $\displaystyle\int_0^1 \frac{x^3}{\sqrt{4+x^2}}\ dx$
3. $\displaystyle\int_1^{\sqrt{3}} \arctan(1/x)\ dx$
4. $\displaystyle\int_4^9 \frac{\ln y}{\sqrt{y}}\ dy$
5. $\int_0^{2\pi} t^2 \sin 2t\ dt$
6. $\int_0^t e^s \sin(t-s)\ ds$

Exercise 5. Use substitution and integration by parts to evaluate each integral.

1. $\int \cos \sqrt{x}\ dx$
2. $\int x \ln(1+x)\ dx$
3. $\int \sin \ln x\ dx$
4. $\int_0^{\pi} e^{\cos x} \sin 2x\ dx$

Exercise 6. Prove that (the reduction formula) $\displaystyle\int (\ln x)^n\ dx = x(\ln x)^n - n \int (\ln x)^{n-1}\ dx.$

6.3 Trigonometric integrals (using trigonometric identities)

Exercise 7. Evaluate the integral. (Use trigonometric identities as needed.)

1. $\int \sin^2 x\ dx$
2. $\int \cos^3 x\ dx$
3. $\int \sin^5 x \cos^2 x\ dx$
4. $\int \sin^4 x\ dx$
5. $\int \sin^2 x \cos^4 x\ dx$
6. $\int \tan^3 x\ dx$
7. $\int \sec^3 x\ dx$
8. $\int \tan^4 x \sec^6 x\ dx$
9. $\int \sin^2 x \cos^3 x\ dx$
10. $\int_0^{\frac{\pi}{4}} \tan^4 x\ dx$
11. $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x\ dx$
12. $\int \sin 4x \cos 5x\ dx$

Exercise 8. Let $m, n \in \mathbb{N}$. Show that

(i) $\displaystyle\int_{-\pi}^{\pi} \sin mx \cos nx\ dx = 0$

(ii) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx\ dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

(iii) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx\ dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

6.4 Trigonometric substitution

Exercise 9. Use the given trigonometric substitution to evaluate each integral.

1. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}} \quad (x = 2\sin\theta)$
2. $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\ dx \quad (x = 2\tan\theta)$
3. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\ dx \quad (x = 2\sec\theta)$

Exercise 10. Evaluate the integral using suitable trigonometric substitution.

1. $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\ dx$
2. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\ dx$
3. $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\ dx$
4. $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\ dx$
5. $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\ dx$
6. $\int_0^1 \sqrt{1+x^2}\ dx$
7. $\displaystyle\int \frac{dx}{\sqrt{16+x^2}}$
8. $\displaystyle\int \frac{du}{u\sqrt{5-u^2}}$
9. $\int \sqrt{5+4x-x^2}\ dx$
10. $\displaystyle\int \frac{x}{\sqrt{1+x+x^2}}\ dx$

6.5 Integration by partial fractions

Exercise 11. Evaluate each integral.

1. $\displaystyle\int \frac{x^3+x}{x-1}\,dx$
2. $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$
3. $\displaystyle\int \frac{1}{x^2-9}\,dx$
4. $\displaystyle\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx$
5. $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx$
6. $\displaystyle\int \frac{x^4}{x-1}\,dx$
7. $\displaystyle\int \frac{3x-2}{x+1}\,dx$
8. $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx$
9. $\displaystyle\int \frac{x^5+x-1}{x^3+1}\,dx$
10. $\displaystyle\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx$

6.6 Other methods

∘ Using antiderivatives (when an antiderivative is known) [see 5.1]

∘ Using areas (when area is known) [see 5.2 (Ex.9)]

∘ Using symmetry (when symmetry is available for odd or even functions)

Exercise 12. Evaluate (a) $\displaystyle\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}$ (b) $\int_{-2}^{2}(x^6+1)\,dx$.

∘ Using numerical methods (when analytical method is unknown or formula is unknown) [see 5.5]