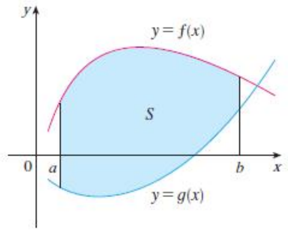
7 Applications of Integration

7.1 Finding Areas

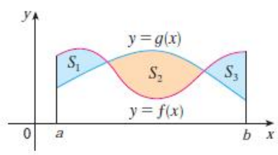


∘ The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, and $x = b$ where f and g are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Exercise 1. Sketch the region bounded above by $y = x^2 + 1$, bounded below by $y = x$ and find the area of the region.

Exercise 2. Sketch the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$ and find the area of the region.



∘ The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, and $x = b$ where f and g are continuous for all $x \in [a, b]$, is

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Exercise 3. Sketch the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{2}$. Find the area of the region.

Exercise 4. Sketch the region enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. Find the area of the region.

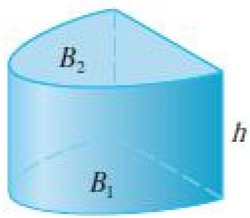
Exercise 5. Sketch the region bounded by the curves $y = \cos x$, $y = \sin 2x$, $x = 0$, and $x = \frac{\pi}{2}$. Find the area of the region.

Exercise 6. Sketch the region enclosed by the the parabolas $x = 1 - y^2$ and $x = y^2 - 1$. Find the area of the region.

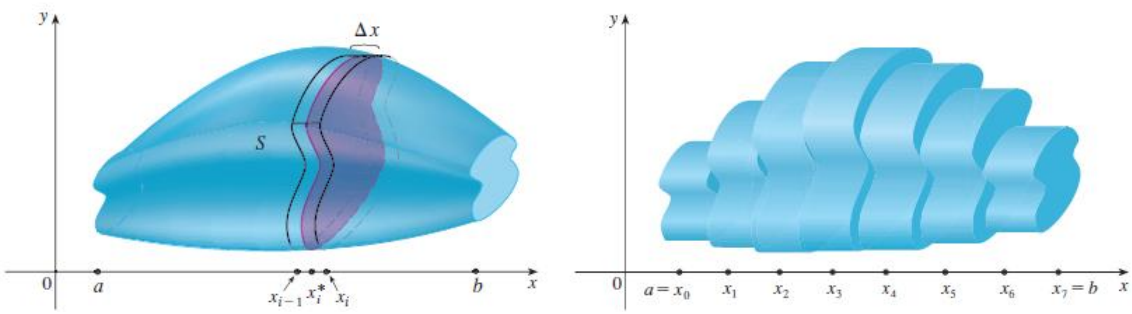
Exercise 7. Find the values of a such that the area of the region bounded by the parabolas $y = x^2 - a^2$ and $y = a^2 - x^2$ is 576.

Exercise 8. Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

7.2 Finding Volumes



∘ A cylinder is bounded by a plane region B_1, called the *base*, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments that are perpendicular to the base and join B_1 to B_2. If the area of the base is A and the height of the cylinder (the distance from B_1 to B_2) is h, then the volume of the cylinder is defined as $V = Ah$.



(***Slab method***) Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{\infty} A(x^\star)\delta x = \int_a^b A(x)\, dx.$$

Exercise 9. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Exercise 10. (***Disk method***) Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

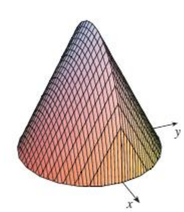
Exercise 11. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, $x = 0$ and about the y-axis.

Exercise 12. (***Washer method***) The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

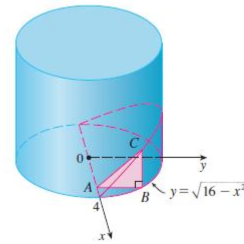
Exercise 13. The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $y = 2$. Find the volume of the resulting solid.

Exercise 14. The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $x = -1$. Find the volume of the resulting solid.

Exercise 15. Figure shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



Exercise 16. A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.



Exercise 17. Find the volume of the solid obtained by rotating the region bounded by the curves $y = 1 - x^2$, and $y = 0$ about the x-axis. Sketch the region, the solid, and a typical disk or washer.

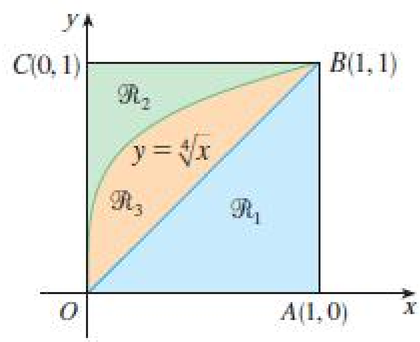
Exercise 18. Find the volume of the solid obtained by rotating the region bounded by the curves $x = 2\sqrt{y}$, $x = 0$, and $y = 9$ about the y-axis. Sketch the region, the solid, and a typical disk or washer.

Exercise 19. Find the volume of the solid obtained by rotating the region bounded by the curves $x = y^2$,and $y = x^2$ about the line $y = 1$.

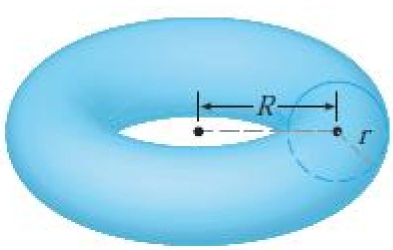
Exercise 20. Find the volume of the solid obtained by rotating the region bounded by the curves $x = y^2$,and $y = x^2$ about the line $x = -1$.

Exercise 21. Refer to the figure and find the volume generated by rotating the given region about the specified line.

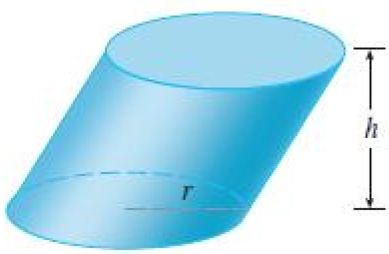
(a) $\mathscr{R}_1$ about OA.	(e) $\mathscr{R}_2$ about OA.	(i) $\mathscr{R}_3$ about OA.
(b) $\mathscr{R}_1$ about OC.	(f) $\mathscr{R}_2$ about OC.	(j) $\mathscr{R}_3$ about OC.
(c) $\mathscr{R}_1$ about AB.	(g) $\mathscr{R}_2$ about AB.	(k) $\mathscr{R}_3$ about AB.
(d) $\mathscr{R}_1$ about BC.	(h) $\mathscr{R}_2$ about BC.	(l) $\mathscr{R}_3$ about BC.

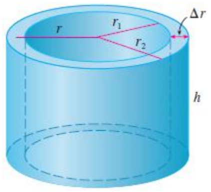


Exercise 22. Find the volume of torus with radii r and R as shown in figure.



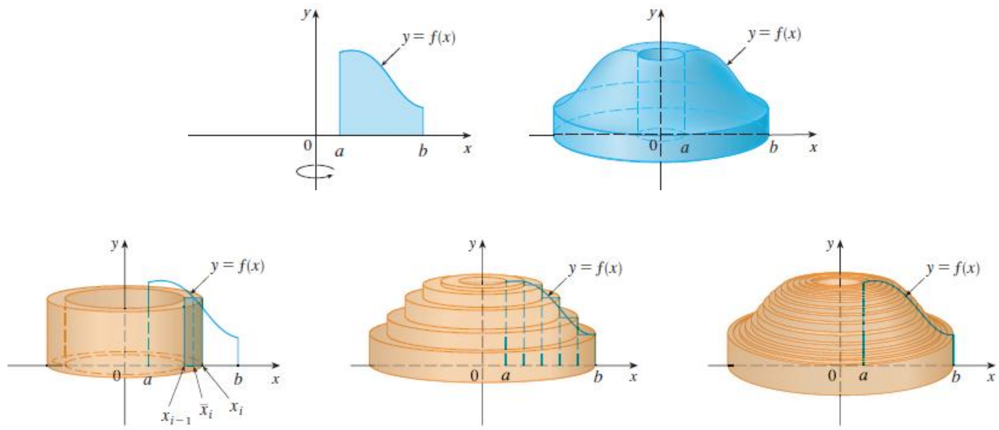
Exercise 23. find the volume of the oblique cylinder shown in the figure.





○ Figure shows a cylindrical shell with inner radius r_1, outer radius r_2, and height h. Its volume is calculated by subtracting the volume of the inner cylinder from the volume of the outer cylinder: $V = 2\pi r h \delta r$ where $\delta r = r_2 - r_1$ and $r = \frac{1}{2}(r_1 + r_2)$.

(***Shell method***) Let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ where $(f(x) \geq 0)$ $y = 0$, $x = a$, and $x = b$, where $b > a \geq 0$.



The volume of the solid S,

$$V = \lim_{n\to\infty} \sum_{i=1}^{\infty} 2\pi x_i^{\star} f(x_i^{\star}) \delta x = \int_a^b 2\pi x f(x)\, dx.$$

Exercise 24. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Exercise 25. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

Exercise 26. Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Exercise 27. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

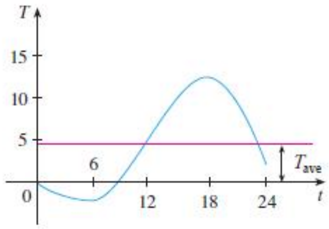
Exercise 28. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves $y = x^2$ and $y = 6x - 2x^2$ about the y-axis.

Exercise 29. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves $xy = 1$, $x = 0$, $y = 1$ and $y = 3$ about the x-axis.

Exercise 30. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves $y = 3$ and $y = 4x - x^2$ about the line $x = 1$.

Exercise 31. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves $x = y^2 + 1$ and $x = 2$ about the line $y = -2$.

7.3 Mean value of a function



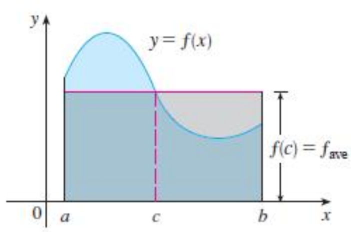
Let f be a continuous function on $[a, b]$. The *average value* of f on the interval $[a, b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\ dx.$$

Exercise 32. Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Exercise 33. Find the average value of the function $f(x) = \sin 4x$ on the interval $[-\pi, \pi]$.

Exercise 34. The linear density in a rod $8m$ long is $12/\sqrt{x+1}$Kg/m, where x is measured in meters from one end of the rod. Find the average density of the rod.



The Mean Value Theorem for Integrals:
Let f be a continuous function on $[a, b]$. Then there exists a number $c \in [a, b]$ such that

$$f_{avg} = f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx.$$

Exercise 35. If f is continuous and $\int_1^3 f(x)\ dx = 8$ then show that f takes on the value 4 at least once on the interval $[1, 3]$.

Let f be a continuous function on $[a, b]$.

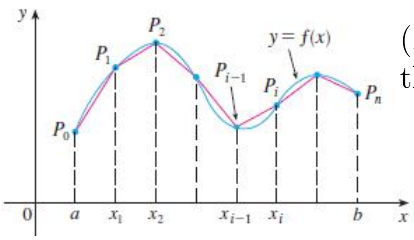
The *mean square value* of $f = \frac{1}{b-a}\int_a^b [f(x)]^2\ dx$.

The *root mean square value* of $f = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\ dx}$.

Exercise 35. Find the root mean square (rms) value of the function $f(x) = x^4$ on the interval $[1, 2]$.

Exercise 36. Find the root mean square (rms) value of the function $f(x) = \sin x$ on the interval $[0, 2\pi]$.

7.4 Arc Length



(*Arc length formula*) Let f' be a continuous function on $[a,b]$. Then then the length of the curve $y = f(x)$, $a \leq x \leq b$ is:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Exercise 37. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1,1) and (4,8).

Exercise 38. Find the length of the arc of the curve $y = 1 + 6x^{3/2}$, $0 \leq x \leq 1$.

If a curve has the equation $x = g(y)$, $c \leq y \leq d$ and $g'(y)$ be a continuous function on $[c,d]$. Then the length of the curve $x = g(y)$, $c \leq y \leq d$ is:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Exercise 39. Find the length of arc given by $x = \frac{2}{3}(y-1)^{3/2}$ where $1 \leq y \leq 4$.

Exercise 40. Find the length of arc given by $x = \frac{1}{3}\sqrt{y}(y-3)$ where $1 \leq y \leq 9$.

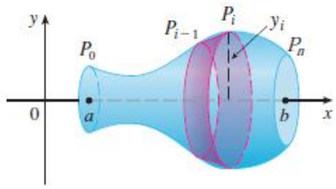
(*Arc length function*) If a smooth curve $\mathscr{C}$ has the equation $y = f(x)$, $a \leq x \leq b$, let $s(x)$ be the distance along $\mathscr{C}$ from the initial point $(a, f(a))$ to the point $(x, f(x))$. Then $s(x)$ is a function, called the arc length function, and :

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

Exercise 41. Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ starting at the point $(1,1)$.

Exercise 42. Find the arc length function for the curve $y = 2x^{3/2}$ with starting point (1,2).

7.5 Area of a Surface of Revolution

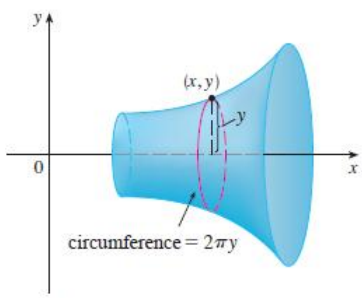


Suppose f is positive and f has a continuous derivative. Then the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis is:

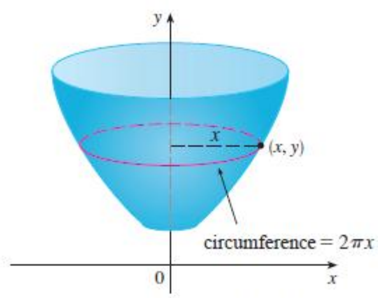
$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

If the curve is given by $x = g(y)$, $c \leq y \leq d$, then

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$



(a) Rotation about x-axis: $S = \int 2\pi y\, ds$



(b) Rotation about y-axis: $S = \int 2\pi x\, ds$

$$ds = \sqrt{1 + \left(\tfrac{dx}{dy}\right)^2}\, dy$$

$$ds = \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx$$

Exercise 43. The curve $y = \sqrt{4 - x^2}$, $-1 \leq x \leq 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis.

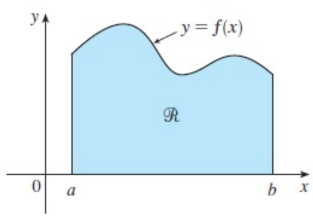
Exercise 44. Find the exact area of the surface obtained by rotating the curve $y = x^3$, $0 \leq x \leq 2$ about the x-axis.

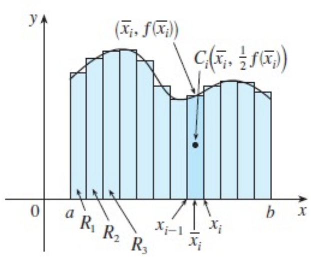
Exercise 45. The arc of the parabola $y = x^2$ from (1,1) to (2,4) is rotated about the y-axis. Find the area of the resulting surface.

Exercise 46. Find the exact area of the surface obtained by rotating the curve $y = 1 - x^2$, $0 \leq x \leq 1$ about the y-axis.

7.7 Moments and Centers of Mass

The point P on which a thin plate of any given shape balances horizontally is called the *center of mass* (or centroid or center of gravity) of the plate.





Consider a flat plate (called a *lamina*) with *uniform density* ρ that occupies a region $\mathcal{R}$ of the plane.

The moment of $\mathcal{R}$ about the y-axis, M_y:

$$M_y = \lim_{n\to\infty} \sum_{n=1}^{n} \rho \overline{x}_i f(\overline{x}_i)\Delta x = \rho \int_a^b x f(x)\ dx.$$

The moment of $\mathcal{R}$ about the x-axis, M_x:

$$M_x = \lim_{n\to\infty} \sum_{n=1}^{n} \rho \frac{1}{2}[f(\overline{x}_i)]^2 \Delta x = \rho \int_a^b \frac{1}{2}[f(x)]^2\ dx.$$

The mass of the plate, m:

$$m = \rho \int_a^b f(x)\ dx.$$

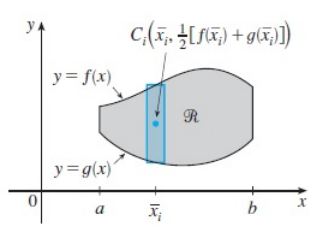
The center of mass of the plate, $(\overline{x}, \overline{y})$:

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

I.e., $\overline{x} = \frac{1}{A}\int_a^b x f(x)\ dx, \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\ dx$ where $A = \int_a^b f(x)\ dx$.

Exercise 47. Find the center of mass of a semicircular plate of radius r.

Exercise 48. Find the centroid of the region bounded by the curves $y = \cos x, y = 0, x = 0$, and $x = \frac{\pi}{2}$.



If the region $\mathcal{R}$ lies between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ as shown in figure then the centroid of $\mathcal{R}$, $(\overline{x}, \overline{y})$ is given by:

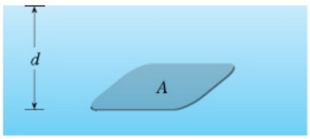
$$\overline{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\ dx, \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\ dx$$

where $A = \int_a^b [f(x) - g(x)]\ dx$.

Exercise 49. Find the centroid of the region bounded by the line $y = x$, and parabola $y = x^2$.

Exercise 50. Find the centroid of the region bounded by the curves $y = 2 - x^2$, and $y = x$.

7.8 Hydrostatic Pressure, Force, and Work Done



Suppose that a thin horizontal plate with area A square meters is submerged in a fluid of density ρ kilograms per cubic meter at a depth d meters below the surface of the fluid as shown in figure. The force F exerted by the fluid on the plate is:

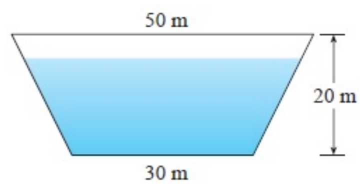
$$F = \rho g A d$$

where is the acceleration due to gravity. The pressure P on the plate is defined to be the force per unit area:

$$P = \rho g d.$$

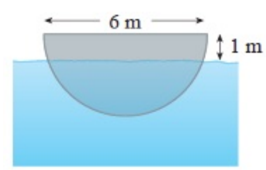
At any point in a liquid the pressure is the same in all directions.

Exercise 51. A dam has the shape of the trapezoid shown in Figure. The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.



Exercise 52. Find the hydrostatic force on one end of a cylindrical drum with radius 3 ft if the drum is submerged in water 10 ft deep.

Exercise 53. A vertical plate is partially submerged in water and has the indicated shape. Explain how to approximate the hydrostatic force against one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it.



∘ If the force is constant the *work done*is defined to be the product of the force and the distance that an object moves: $W = Fd$.

Exercise 54. When a particle is located a distance xm from the origin, a force of $x^2 + 2x$ Newton acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

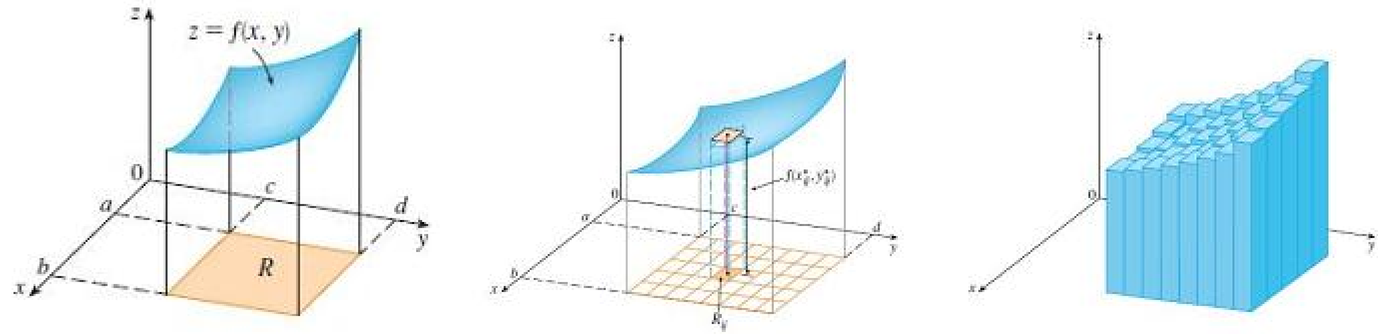
Exercise 55. A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm? (Hooke's Law states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x.)

Exercise 56. A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

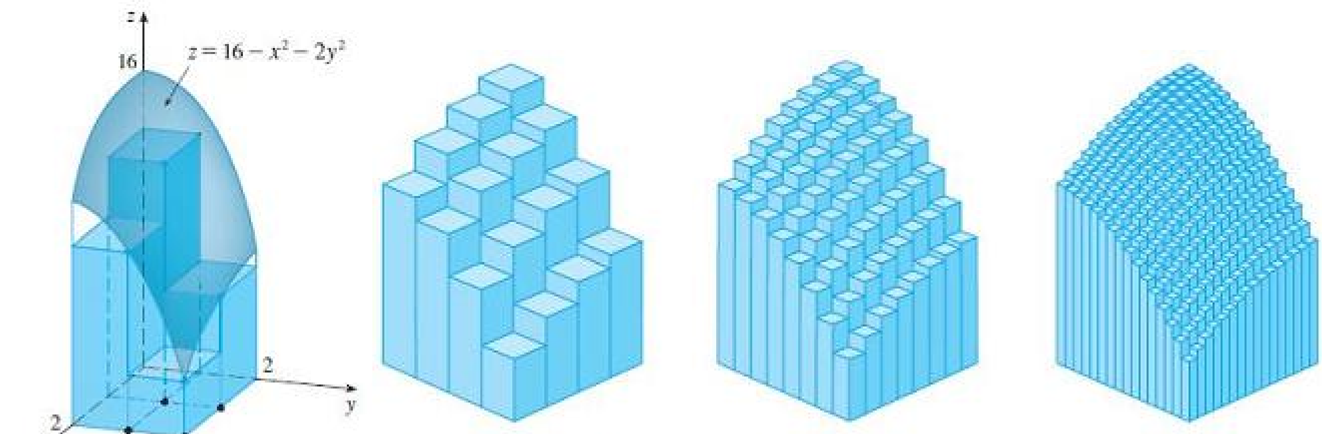
Exercise 57. A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)

Exercise 58. A chain lying on the ground is 10 m long and its mass is 80 kg. How much work is required to raise one end of the chain to a height of 6 m?

7.9 Double Integrals

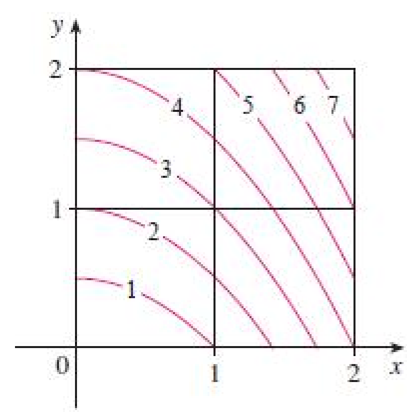


Exercise 59. Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$.



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Exercise 60. The figure (contour map) shows level curves (contour curves) of a function f in the square $R = [0, 2] \times [0, 2]$. Use the Midpoint Rule with $m = n = 2$ to estimate $\int\int_R f(x, y)\, dA$. How could you improve your estimate? Estimate the average value of f.



Exercise 61. Evaluate the iterated integrals. (a) $\displaystyle\int_0^3 \int_1^2 x^2 y \; dydx$ (b) $\displaystyle\int_1^2 \int_0^3 x^2 y \; dxdy$