1 Sequences and Series

1.1 Sequences

∘ A sequence is a list of numbers written in a given order, denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Exercise 1. List the first 10 terms in the sequence given by $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-2} + a_{n-1}$ for all $n \geq 3$.

Exercise 2. Find a formula for the nth term a_n of the sequence $\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \cdots\right\}$.

Definition: A sequence $\{a_n\}$ ***converges*** if $\lim_{n\to\infty} a_n$ exists. Otherwise the sequence ***diverges***.

Exercise 3. Does the *harmonic sequence* $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ converge?

Exercise 4. Determine $\{(-1)^n\}$ is convergent or divergent sequence.

Theorem: If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

Exercise 5. Is the *alternating harmonic sequence* $\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots\}$ convergent?

Theorem: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n = L$.

Exercise 6. For which values of r, the sequence $\{r^n\}$ converges?

Theorem: If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Exercise 7. Does the sequence $\{\sin(\frac{\pi}{n})\}$ converge?

Note: The sequence $\{a, a+d, \cdots, a+(n-1)d, \cdots\}$ is known as arithmetic sequence. The sequence $\{a, ar, ar^2 \cdots, ar^{(n-1)}, \cdots\}$ is known as geometric sequence.

1.2 Series

Definition: Let $\{a_n\}_{n=1}^{\infty}$ be a given sequence. The sum of terms $a_1 + a_2 + \cdots + a_n + \cdots$, denoted by $\sum_{n=1}^{\infty} a_n$, is called an ***(infinite) series***.

Definition: Given a series $\sum_{n=1}^{\infty} a_n$, let S_n denotes the nth partial sum $S_n = \sum_{n=1}^{\infty} a_n$. If the sequence $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = S$ then we say the series $\sum_{n=1}^{\infty} a_n$ is convergent and call S the sum of the series. If the sequence $\{S_n\}$ is divergent then we say the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Exercise 8. Show that the *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Exercise 9. Show that the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

Exercise 10. Show that the *alternating series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

Exercise 11. For what values of r the series $\sum_{n=1}^{\infty} ar^{n-1}$ converges? $(a \neq 0)$

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

Exercise 12. Determine the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ converges or diverges.

1.3 Tests for convergence

1.3.1 The integral test

Suppose f is continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then,

- If $\int_1^\infty f(x)\, dx$ is convergent then $\sum_{n=1}^\infty a_n$ is convergent.
- If $\int_1^\infty f(x)\, dx$ is divergent then $\sum_{n=1}^\infty a_n$ is divergent.

Exercise 13. Test for convergence: $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+1}$.

Exercise 14. Show that the p-series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

1.3.2 The comparison test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ also divergent.

Exercise 15. Test for convergence: $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$.

Exercise 16. Test for convergence: $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$.

1.3.3 Alternating series test

If the alternating series $\sum (-1)^{n-1} b_n$, $(b_n > 0)$ satisfies (i) $b_{n+1} \leq b_n$ for all n and (ii) $\lim\limits_{n\to\infty} = 0$ then the series is convergent.

Exercise 17. Test for convergence: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

Exercise 18. Test for convergence: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+1}$.

1.3.4 Ratio test

Definition: A series $\sum a_n$ is called ***absolutely convergent*** if $\sum |a_n|$ is convergent.

Exercise 19. Test for absolute convergence: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$.

Exercise 20. Test for absolute convergence: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

Definition: A series $\sum a_n$ is called ***conditionally convergent*** if it is convergent but not absolutely convergent.

Exercise 21. Test for conditional convergence: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

Theorem: If $\sum a_n$ is absolutely convergent then it is convergent.

Exercise 22. Test for convergence: $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

<u>The ratio test</u>

Let $\sum a_n$ be given.

- ∘ If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum a_n$ is absolutely convergent.
- ∘ If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$ then $\sum a_n$ is divergent.
- ∘ If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then the test is inconclusive.

Exercise 22. Test for absolute convergence: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$.

Exercise 23. Test for convergence: $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

1.4 Power series

Definition: A ***power series*** is a series of the form $\sum_{n=0}^{\infty} c_n x^n$ where x is a variable and c_n are constants call the coefficients of the power series. The function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has domain as the set of all x for which the series converges. The series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is called power series centered at a.

Exercise 24. For which values of x the series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ converges?

Theorem: For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only 3 possibilities.

∘ The series converges only when $x = a$.

∘ The series converges for all $x \in \mathbb{R}$.

∘ The series converges for $|x-a| < R$ and diverges for $|x-a| > R$ for some $R > 0$ known as *radius of convergence.*

Note: Interval of convergence depends on the behavior of series at the end points.

Exercise 25. Find the radius of convergence and interval of convergence: $\sum_{n=0}^{n=\infty} \frac{(-3)^n x}{\sqrt{n+1}}$.

Exercise 26. Express $f(x) = \frac{1}{1-x}$ as a power series.

Exercise 27. Express $f(x) = \frac{1}{1+x^2}$ as a power series.

Theorem: Let $f(x) = \sum_{n=0}^{n=\infty} c_n(x-a)^n$ with radius of convergence $R > 0$. Suppose f is differentiable. Then,

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \text{ and } \int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}$$

within the same radius of convergence R.

Exercise 28. Express $f(x) = \frac{1}{(1-x)^2}$ as a power series.

Exercise 29. Express $f(x) = \tan^{-1} x$ as a power series.

1.5 Taylor series

Theorem: If f has a power series expansion at a then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, $|x-a| < R$ and this is known as ***Taylor series***. When $a = 0$ this is known as ***Maclaurin series***.

Exercise 30. Find Maclaurin series of $f(x) = e^x$. What is the radius of convergence?

Exercise 31. Find Maclaurin series of $f(x) = \ln(x+1)$. What is the radius of convergence?

Exercise 32. Find Maclaurin series of $f(x) = \sin x$. What is the radius of convergence?

Exercise 33. Find Taylor series of $f(x) = \cos x$ about $x = \frac{\pi}{2}$. What is the radius of convergence?

Note: Let $k \in \mathbb{R}$ and $|x| < 1$. Then $(1+x)^k = \sum ({}_kC_n)x^n = 1+kx+\dfrac{k(k-1)}{2!}x^2+\dfrac{k(k-1)(k-2)}{3!}x^2 \cdots +$ is known as *binomial series*.