4 Applications of Differentiation

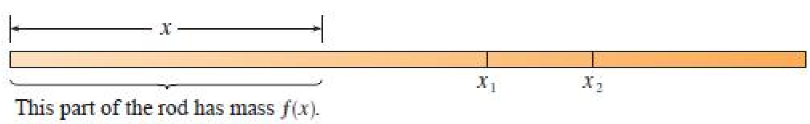
4.1 Rates of change

☐ If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\frac{\Delta s}{\Delta t}$ represents the *average velocity* over a time period Δt, and $v(t) = s'(t)$ represents the *instantaneous velocity* (the rate of change of displacement with respect to time). The instantaneous rate of change of velocity with respect to time is *acceleration*, given by $a(t) = v'(t) = s''(t)$.

Exercise 1. The position of a particle is given by the equation $s(t) = t^3 - 6t^2 + 9t$ where t is measured in seconds and s in meters.

(a) Find the velocity at time t.
(b) What is the velocity after 2 s? After 4 s?
(c) When is the particle at rest?
(d) When is the particle moving forward (that is, in the positive direction)?
(e) Draw a diagram to represent the motion of the particle.
(f) Find the total distance traveled by the particle during the first five seconds.
(g) Find the acceleration at time t and after 4 s.
(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.
(i) When is the particle speeding up? When is it slowing down?

☐ If a rod or piece of wire is homogeneous, then its *linear density* is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous (i.e., non-homogeneous) but that its mass measured from its left end to a point x may be given by a function $f(x)$.



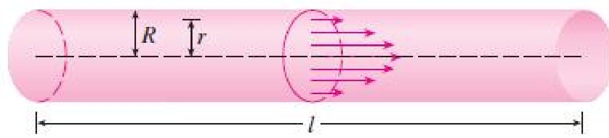
Exercise 2. Suppose the mass (m) of a rod varies according to the function $m = f(x) = \sqrt{x}$ where $x = 0$ point corresponds to the left end point of the rod where is x measured in meters and m in kilograms.

(a) Find the average density of the part of the rod given by $1 \leq x \leq 1.2$.
(b) Find the density when $x = 1$.

☐ A current exists whenever electric charges move. If δQ is the net charge that passes through a given surface during a time period Δt, then the *average current* during this time interval is defined as $\frac{\Delta Q}{\Delta t}$. The current $I = \frac{dQ}{dt}$ is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes).

Exercise 3. The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when (a) $t = .5s$ and (b) $t = 1s$. [The unit of current is an ampere ($1A = 1C/s$).] At what time is the current lowest?

☐ When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius R and length l. The velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall. The relationship between v and r is given by the *law of laminar flow* $v = \frac{P}{4\eta l}(R^2 - r^2)$ where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$. The instantaneous rate of change of velocity with respect to r, $\frac{dv}{dr}$ is known as the *velocity gradient.*



Exercise 4. For one of the smaller human arteries; $\eta = .027$, $R = .008cm$, $l = 2cm$, and $P = 4000 dynes/cm^2$. Find the velocity gradient at $r = .002cm$.

☐ One of the quantities of interest in thermodynamics is *compressibility.* If a given substance is kept at a constant temperature, then its volume V depends on its pressure P. We can consider the rate of change of volume with respect to pressure, namely, the derivative $\frac{dV}{dP}$. The compressibility is defined by introducing a minus sign and dividing this derivative by the volume V: isothermal compressibility $= \beta = -\frac{1}{V}\frac{dV}{dP}$.

Exercise 5. The volume V (in cubic meters) of a sample of air at 25° was found to be related to the pressure P (in kilopascals) by the equation $V = \frac{5.3}{P}$. Find the compressibility when the pressure is 50kPa.

Exercise 6. A spherical balloon is being inflated. Find the rate of increase of the surface area with respect to the radius r when r is (a) 1 ft, (b) 2 ft, and (c) 3 ft. What conclusion can you make?

Exercise 7. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant: $PV = C$.
(a) Find the rate of change of volume with respect to pressure.
(b) A sample of gas is in a container at low pressure and is steadily compressed at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes? Explain.

Exercise 8. The frequency of vibrations of a vibrating violin string is given by $f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$ where L is the length of the string, T is its tension, and ρ is its linear density. Find the rate of change of the frequency with respect to
(a) the length (when T and ρ are constant),
(b) the tension (when ρ and L are constant), and
(c) the linear density (when L and T are constant).

Exercise 9. The total cost that a company incurs in producing x units is known as the *cost function.* The cost function for production of a commodity is $c(x) = 339 + 25x - .09x^2 + .0004x^3$.
(a) Find $c'(100)$.
(b) Interpret $c'(100)$.
(c) Find the cost of producing 101st item.
(d) Compare $c'(100)$ with answer in (c).

Exercise 10. In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP}{dt} = r_0\left(1 - \frac{P(t)}{P_c}\right)P(t) - \beta P(t)$$

where r_0 is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the carrying capacity), and β is the percentage of the population that is harvested. If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.

4.2 Related Rates

☐ In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Exercise 11. Air is being pumped into a spherical balloon so that its volume increases at a rate of $100cm^3/s$. How fast is the radius of the balloon increasing when the diameter is 50cm?

Exercise 12. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Exercise 13. A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of $2m^3$/min, find the rate at which the water level is rising when the water is 3m deep.

Exercise 14. Car A is traveling west at 50mi/h and car B is traveling north at 60mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

Exercise 15. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

Exercise 16. The radius of a sphere is increasing at a rate of 4mm/s. How fast is the volume increasing when the diameter is 80 mm?

Exercise 17. A particle is moving along a hyperbola $xy = 8$. As it reaches the point (4,2), the $y-$coordinate is decreasing at a rate of 3cm/s. How fast is the $x-$coordinate of the point changing at that instant?

Exercise 18. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30 cm deep?

4.3 Linear Approximations and Differentials

4.3.1 Linearizations

We use the tangent line at $(a, f(a))$ as an approximation to the curve when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

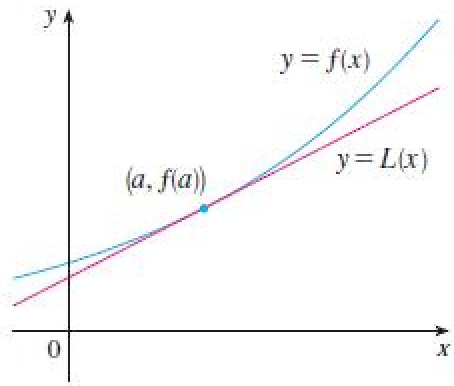
and the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the *linear approximation* or *tangent line approximation* of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the *linearization* of f at a.

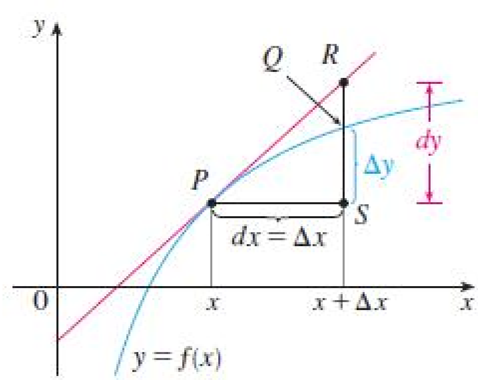


Exercise 19. Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

Exercise 20. For what values of is the linear approximation $\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$ accurate to within 0.5? What about accuracy to within 0.1?

4.3.2 Differentials

If $y = f(x)$, where f is a differentiable function, then the *differential* dx is an independent variable; that is, dx can be given the value of any real number. The differential dy is then defined in terms of dx by the equation $dy = f'(x)dx$. So dy is a dependent variable; it depends on the values of x and dx. If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined.



Exercise 21. Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

Exercise 22. The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere? What is the relative error? What is the percentage error?

Exercise 23. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

Exercise 24. If a current passes through a resistor with resistance R, Ohm's Law states that the voltage drop is $V = RI$. If V is constant and R is measured with a certain error, use differentials to show that the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

Exercise 25. Verify the linear approximation $(1 + x)^{-3} \approx 1 - 3x$ at $a = 0$. Then determine the values of for which the linear approximation is accurate to within 0.1.

Exercise 26. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

4.4 Exponential Growth and Decay

☐ In many natural phenomena, quantities grow or decay at a rate proportional to their size. That is, $f'(t) = kf(t)$ for some constant k. In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then $\frac{dy}{dt} = ky$ where is a constant. This is sometimes called the *law of natural growth* (if $k > 0$) or the *law of natural decay* (if $k < 0$). It is called a *differential equation* because it involves an unknown function y and its derivative $y'(t)$.

Theorem: The only solutions of the differential equation $\frac{dy}{dt} = ky$ are the exponential functions $y(t) = y(0)e^{kt}$.

☐ When $k > 0$ the quantity $(\frac{dy}{dt})/y$ is known as the *relative growth rate.* The time taken a quantity to double the initial amount known as *doubling time.*

Exercise 27. Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

☐ When $k < 0$ the time required for half of any given quantity to decay is known as *half-life.*

Exercise 28. The half-life of radium-226 is 1590 years.
(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.
(b) Find the mass after 1000 years correct to the nearest milligram.
(c) When will the mass be reduced to 30 mg?

☐ **Newton's Law of Cooling:** Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming.) If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s) \text{ where } k \text{ is a constant.}$$

Exercise 29. A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F.
(a) What is the temperature of the soda pop after another half hour?
(b) How long does it take for the soda pop to cool to 50°F?

☐ Scientists can determine the age of ancient objects by the method of *radiocarbon dating.* The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.

Exercise 30. A parchment fragment was discovered that had about 74% as much ^{14}C radioactivity as does plant material on the earth today. Estimate the age of the parchment.

4.5 Newton's Method

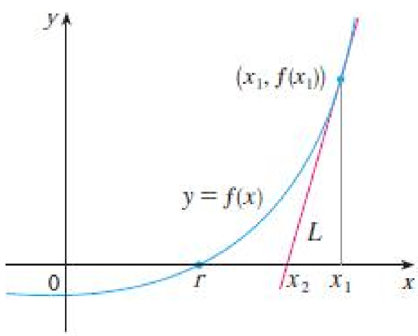
☐ Suppose f is a differentiable function and we want to solve $f(x) = 0$ for x. When this is not possible or difficult to solve analytically we will use numerical root finding method. One such method is Newton's method (or Newton-Raphson method).

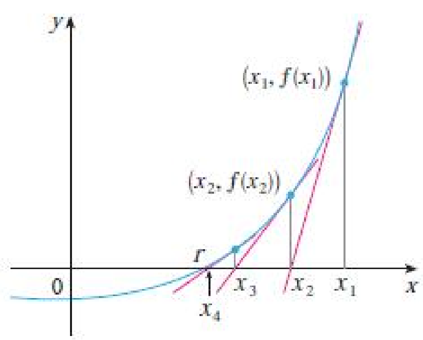
☐ We start with a first approximation x_1 (which is often a guess), and find the tangent line to the curve at $(x_1, f(x_1))$, and find the $x-$intercept of the tangent line and name it x_2 the second approximation. (See first figure below.)

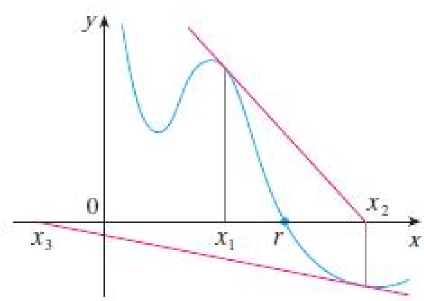
☐ We repeat this process until we obtain a good approximation to the root r of $f(x) = 0$. (see second figure below.) In this iterative process we get a sequence of approximations $x_1, x_2, x_3, \cdots, x_n$ and if the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence converges to r and we write $\lim\limits_{n\to\infty} x_n = r$.

☐ In practice, we stop the process when the desired accuracy if achieved. The rule of thumb that is generally used is that we can stop when successive approximations and agree to desired number of decimal places.

☐ In certain situations the sequence may not converge and in such situations a better initial approximations should be chosen. (See third figure below.)







$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{where } f'(x_n) \neq 0.}$$

Exercise 31. Starting with $x_1 = 2$ find a third approximation x_3 to the (real) root of $x^3 - 2x - 5 = 0$.

Exercise 32. Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

Exercise 33. Find, correct to six decimal places, the root of the equation $x = \cos x$ using Newton's method.

Exercise 34. Approximate the root of $2.2x^5 - 4.4x^3 + 1.3x^2 - 0.9x - 4 = 0$ correct to 6 decimal places using Newton's method.

Exercise 35. Find all roots of the equation $\sin x = x^2 - 2$ correct to 6 decimal places using Newton's method.

4.6 l'Hospital's Rule

☐ Indeterminate form of type $\frac{0}{0}$: If we have a limit in the form, $\lim_{x \to a} \dfrac{f(x)}{g(x)}$, where both $f(x)$ and $g(x)$ approach 0 as x approaches a, then this limit may or may not exist and called an *indeterminate form* $\frac{0}{0}$.

☐ Indeterminate form of type $\frac{\infty}{\infty}$: If we have a limit in the form, $\lim_{x \to a} \dfrac{f(x)}{g(x)}$, where both $f(x)$ and $g(x)$ approach $\pm\infty$ as x approaches a, then this limit may or may not exist and called an *indeterminate form* $\frac{\infty}{\infty}$.

l'Hospital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty.$$

Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

where if the limit on the right side exists or $\pm\infty$.

Note: The l'Hospital's Rule is valid if $x \to a$ replaced by $x \to a^-$, $x \to a^+$, $x \to +\infty$, or $x \to -\infty$.

Exercise 36. Find $\lim_{x \to 1} \dfrac{\ln x}{x-1}$. ($\frac{0}{0}$ type)

Exercise 37. Find $\lim_{x \to \infty} \dfrac{e^x}{x^2}$. ($\frac{\infty}{\infty}$ type)

Exercise 38. Find $\lim_{x \to 0} \dfrac{\tan x - x}{x^3}$.

☐ Other indeterminate forms are: $0 \cdot \infty$ (indeterminate product), $\infty - \infty$ (indeterminate difference), 0^0, ∞^0, 1^∞ (indeterminate powers). Put these in to 0/0 form or ∞/∞ form and apply l'Hospital's Rule.

☐ Indeterminate form of type $0 \cdot \infty$:

Exercise 39. Find $\lim_{x \to 0^+} x \ln x$.

Exercise 40. Find $\lim_{x \to 0^+} \sin x \ln x$.

☐ Indeterminate form of type $\infty - \infty$:

Exercise 41. Find $\lim_{x \to (\frac{\pi}{2})^-} \sec x - \tan x$.

Exercise 42. Find $\lim_{x \to \infty} x - \ln x$.

☐ Indeterminate powers:

Exercise 43. Find $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

Exercise 44. Find $\lim_{x \to 0^+} x^x$.

Exercise 45. Find $\lim_{x \to 0^+} x^{\frac{1}{x}}$.

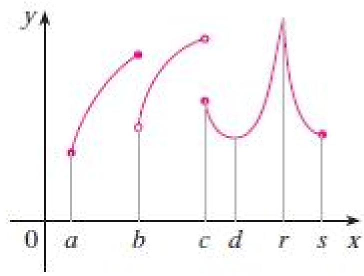
4.7 Maximum and Minimum Values (Extreme Values)

Definition: Let c be a number in the domain D of a function f. Then $f(c)$ is the
☐ *absolute (or global) maximum value* of f on D if $f(c) \geq f(x)$ for all $x \in D$.
☐ *absolute (or global) minimum value* of f on D if $f(c) \leq f(x)$ for all $x \in D$.

The number $f(c)$ is a
☐ *local maximum value* of f if $f(c) \geq f(x)$ when x is near c.
☐ *local minimum value* of f if $f(c) \leq f(x)$ when x is near c.

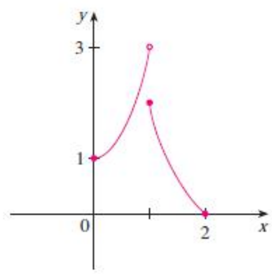
Note: By 'near' we mean x is in some open interval containing point c.

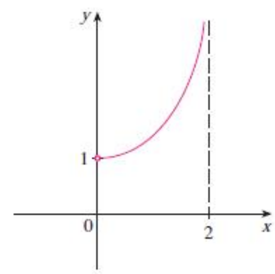
Exercise 46. For each of the numbers $a, b, c, d, r,$ and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.



The Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Exercise 47. Observe that absolute maximum does not exist for both functions given by their graphs. Explain why the extreme value theorem does not contradict the given two graphs.





Fermat's Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Exercise 48. Note that if $f(x) = x^3$ then $f'(0) = 0$ but f has no minimum or maximum. Explain why this does not contradict Fermat's theorem.

Definition: A *critical number* of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Exercise 49. Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

Exercise 50. Find the critical numbers of $g(x) = |3x - 5|$.

Note: If f has a local maximum or minimum at c, then c is a critical number of f.

☐ To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local [in which case it occurs at a critical number] or it occurs at an endpoint of the interval. Thus the following three-step procedure always works.

The Closed Interval Method To find the absolute maximum and minimum values of a continuous function on a closed interval $[a, b]$:
1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

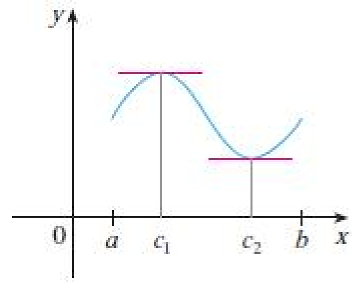
Exercise 51. Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1$, $x \in [-\frac{1}{2}, 4]$.

Exercise 52. Find the absolute maximum and minimum values of the function $g(x) = x + \frac{1}{x}$, $0.2 \leq x \leq 4$.

4.8 The Mean Value Theorem (MVT)

Rolle's Theorem: Let f be a function that satisfies the following three hypotheses:
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then there is a number $c \in (a, b)$ such that $f'(c) = 0$.

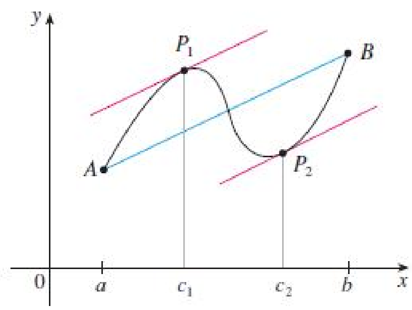


Exercise 53. Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root. (Hint: Apply the IVT first and then Rolle's theorem.)

The Mean Value Theorem: Let f be a function that satisfies the following hypotheses:
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Exercise 54. State true or false; "If a car traveled 180 km in 2 hours, then the speedometer must have read 90 km/h at least once." Justify your answer. (Hint: Apply MVT.)

Exercise 55. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

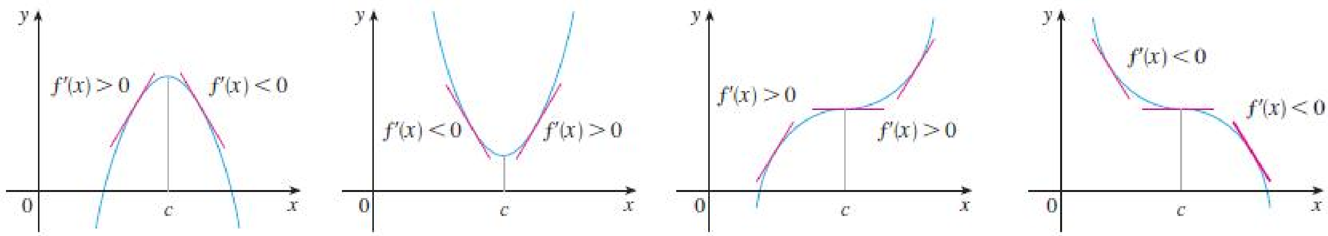
Exercise 56. Prove that if $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

4.9 Curve Sketching

Increasing/Decreasing Test (What f' says about f?)
(a) If $f'(x) > 0$ on an interval, then f is *increasing* on that interval.
(b) If $f'(x) > 0$ on an interval, then f is *decreasing* on that interval.

Exercise 57. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

The First Derivative Test Suppose that c is a critical number of a continuous function f.
(a) If f' changes from positive to negative at c, then f has a local maximum at c.
(b) If f' changes from negative to positive at c, then f has a local minimum at c.
(c) If f' does not change sign at c, then f has no local maximum or minimum at c.

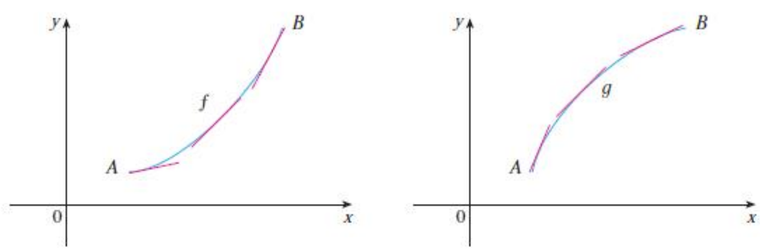


(a) Local maximum (b) Local minimum (c) No maximum or minimum (d) No maximum or minimum

Exercise 58. Find the local minimum and maximum values of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

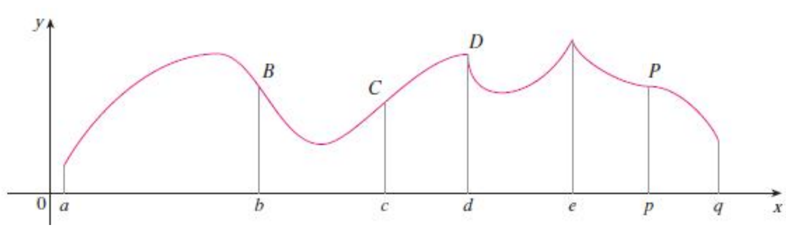
Exercise 59. Find the local maximum and minimum values of the function $f(x) = x + 2\sin x,\ x \in [0, 2\pi]$.

Definition: If the graph of f lies above all of its tangents on an interval I, then it is called *concave upward* (CU) on I. If the graph of f lies below all of its tangents on I, it is called *concave downward* (CD) on I.



(a) Concave upward (b) Concave downward

Exercise 60. Determine on which interval f is concave up or concave down.



Concavity Test (What f'' says about f?)
(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Exercise 61. On which intervals graph of $f(x) = x^4 - 4x^3$ concave upward or concave downward?

Definition: A point P on a curve $y = f(x)$ is called an *inflection point* if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Exercise 62. Determine any inflection points of $f(x) = x^4 - 4x^3$.

The Second Derivative Test: Suppose f'' is continuous near c. (a) If and $f'(c) = 0$, and $f''(c) > 0$ then f has a local minimum at c. (b) If and $f'(c) = 0$, and $f''(c) < 0$ then f has a local maximum at c.

Note: The Second Derivative Test is inconclusive when $f''(c) = 0$.

Exercise 62. Find local minimum or maximum of $f(x) = x^4 - 4x^3$ using second derivative test.

☐ Guidelines to follow when sketching a curve:

1. Identify the domain
2. Check for symmetry (even or odd)
3. Check for periodicity
4. Find intercepts ($x-$ and $y-$ intercepts)
5. Find asymptotes (vertical asymptotes, horizontal asymptotes, or slant asymptotes if any)
6. Intervals of increase or decrease
7. Local maximum or minimum
8. Concavity
9. Inflection points
10. Function values at selected points if needed

Exercise 63. Sketch the graph of the function $f(x) = x^4 - 4x^3$.

Exercise 64. Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

Exercise 65. Sketch the graph of the function $f(x) = \dfrac{2x^2}{x^2 - 1}$.

Exercise 66. Sketch the graph of the function $f(x) = \dfrac{\cos x}{2 + \sin x}$.

Exercise 67. Sketch the graph of the function $f(x) = \dfrac{x^3}{x^2 + 1}$.

4.10 Optimization Problems

Exercise 68. A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Exercise 69. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Exercise 70. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.