Graph Representation

Dr. Jayantha Lanel

University of Sri Jayawardanapura

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Learning Outcomes

- Introduction to data structures for graphs
    1. Drawing graphs and analyzing properties in Graph Theory using **Maple**.

Outline

Introduction to Graph Theory

- Graph Theory is a branch of Discrete Mathematics.
- Graph theory is study of graphs which are mathematical structures used to model pair wise relations between objects.
- A graph is made up of Vertices V (nodes) and edges E (lines) that connect them.

History

- Graph Theory is originated with the problem of *Königsberg bridge*, in 1735. This problem escort to the concept of Eulerian Graph. Euler studied the problem of Köinsberg Bridge and established a structure to resolve the problem called Eulerian graph.
- In 1840, A.F Mobius presented the idea of complete graph and bipartite graph and Kuratowski proved that they are plane by means of recreational problems.
- The concept of tree, (a connected graph without cycles) was enacted by Gustav Kirchhoff in 1845, and he enrolled graph theoretical ideas in the calculation of currents in electrical networks or circuits.
- In 1852, Thomas Gutherie established the famous four color problem. Eventhough the four color problem was invented it was solved only after a century by Kenneth Appel and Wolfgang Haken.

History cont..

- Then in 1856, Thomas.P.Kirkman and William R.Hamilton measured cycles on polyhydra and contrived the concept called Hamiltonion graph by studying trips that visited certain sites exactly once.
- In 1913, H.Dudeney mentioned a puzzle problem.

Quoted by: Imperial Journal of interdisciplinary Research(IJIR), Vol-3, Issue-3, 2017

Königsberg bridge problem

The Seven Bridges of Königsberg is a historically notable problem in mathematics. The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other, or to the two mainland portions of the city, by seven bridges.
Problem: Beginning anywhere and ending anywhere, can a person walk through town crossing all seven bridges but not crossing any bridge twice?

Applications of Graph Theory in Real Life

- GPS or Google maps
  GPS or Google maps are to find a shortest route from one dimension to another. The destinations are vertices and their connections are edges consisting distance.
- Social Networks
  We connect with friends via social media or a video gets viral, here user is a vertex and other connected users create and edge therefore videos get viral when reached to certain connection.
- Using Google to search Web Pages
  Pages are linked to each other by hyperlinks. Each page is a vertex and the link between two pages is an edge.
- We can find lot of examples in computer science, chemistry, biology, physics, operation research, etc...

Definitions

Graph

A *graph* $G = (V, E)$ consists of set V of vertices and a collection E of unordered pairs of vertices called edges.

- Example : $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_3, v_4\}\}$

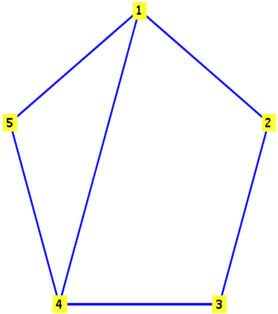
Graph (from Bondy and Murty Text book)

A *graph* G is an ordered triple $(V(G), E(G), \Psi_G)$ consisting of a nonempty set $V(G)$ of *vertices*, a set $E(G)$ of *edges*, and an *incidence function* Ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.

- Example : $G = (V(G), E(G), \Psi_G)$ where $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and Ψ_G is defined by $\Psi_G(e_1) = v_1 v_2, \Psi_G(e_2) = v_1 v_3, \Psi_G(e_3) = v_3 v_4,$ $\Psi_G(e_4) = v_2 v_4, \Psi_G(e_5) = v_1 v_4, \Psi_G(e_6) = v_1 v_1, \Psi_G(e_7) = v_1 v_2$

Example

Draw the graph of $V = \{1, 2, 3, 4, 5\}$ and
$E = \{\{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$
Graph of Example 2 using **Maple**.



Question 1

Draw the graph of $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2\}, \{1, 4\}, \{2, 4\}, \{2, 3\}, \{2, 5\}, \{3, 5\}\}$ using **Maple**.

Definitions cont..

Order

The *order* of a graph is the number of its vertices.

Size

The *size* of a graph is the number of its edges.

Degree

The *Degree* of a vertex is the number of edges adjacent to that vertex.

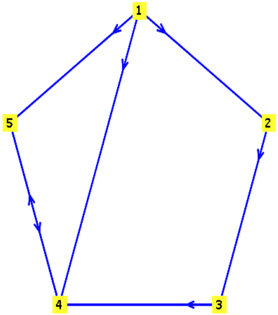
Hand Shaking Lemma

The sum of the degrees of the graph is twice the number of edges in it.

Definitions cont..

Directed Graph or digraph

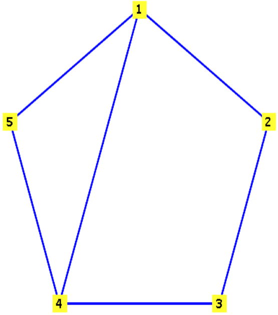
Directed graph have edges with direction. The edges indicate a one-way relationship, in that each edge can only be traversed in a single direction.



Definitions cont..

Undirected Graph

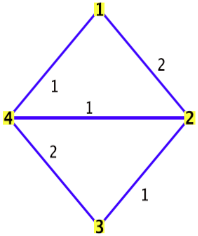
Undirected graph have edges that do not have a direction. The edges indicate a two-way relationship, in that each edge can be traversed in both directions.

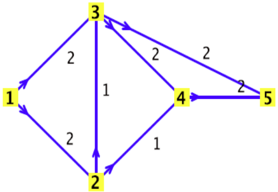


Definitions cont..

Weighted Graph

A *weighted graph* is a graph in which each branch is given a numerical weight.





Question 2

Draw the weighted graph of $V = \{1, 2, 3, 4\}$ and
$E = \{[\{1, 2\}, 2], [\{1, 4\}, 1], [\{2, 4\}, 1], [\{2, 3\}, 1], [\{3, 4\}, 2]\}$ using **Maple**.

Question 3

Draw the weighted digraph of $V = \{1, 2, 3, 4, 5\}$ and
$E = \{[[1, 3], 2], [[1, 2], 2], [[2, 3], 1], [[3, 4], 2], [[2, 4], 1], [[4, 5], 2], [[3, 5], 2]\}$
using **Maple**.

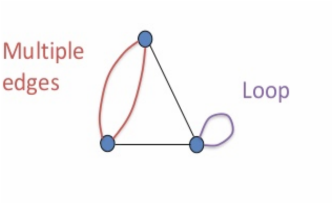
Definitions cont..

Loop

An edge having the same vertex as both its end vertices is called a *loop*.

Parallel edges

Two or more edges that join the same pair of distinct vertices are called *parallel edges*.



Question 4

Draw the graph of $V = \{1, 2, 3, 4, 5, 6\}$ and
$E = \{[1, 3], [1, 2], [2, 3], [3, 4], [2, 2], [4, 5], [4, 5]\}$.

Definitions cont..

Simple Graph

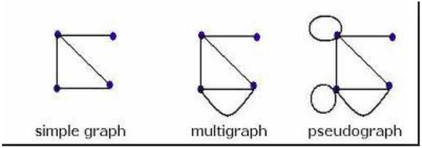
A *simple graph* is a graph with no parallel edges and loops.

Multigraph

A *multigraph* with multiple edges but no loops.

Pseudograph

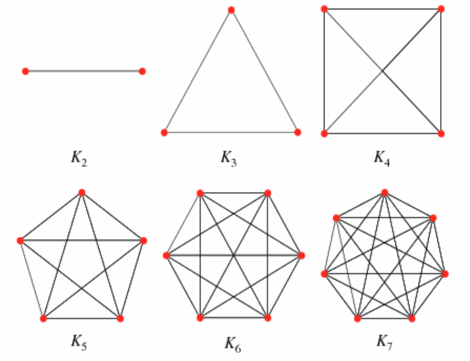
A graph with at least one loop is a *pseudograph*.



Definitions cont..

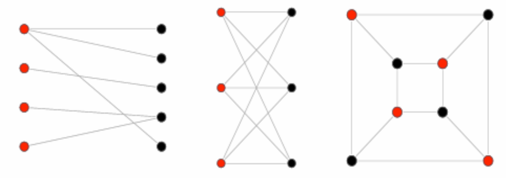
Complete Graph

The K_n is a graph with n vertices said to be *complete graph* if every vertex in K_n is connected to every other vertex in K_n.



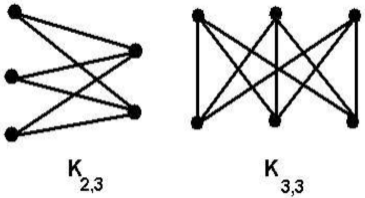
Bipartite Graph

The *bipartite graph* (or *bigraph*) is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V. Vertex sets U and V are usually called the parts of the graph



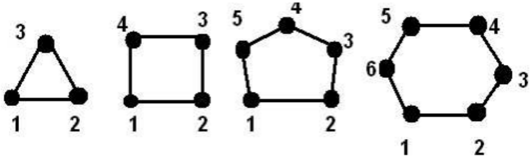
Complete Bipartite Graph $K_{n,m}$

A complete bipartite graph $K_{m,n}$ is a bipartite graph that has each vertex from one set adjacent to each vertex in the other set.



Question 5

Which of the following graphs are bipartite?



Question 6

Find a formula for the number of edges in $K_{n,m}$.

Definitions cont..

Subgraphs

The Graph $H = (W, F)$ is a subgraph of the graph $G = (V, E)$ if W is a subset of V and F is a subset of E.

Isomorphic Graphs

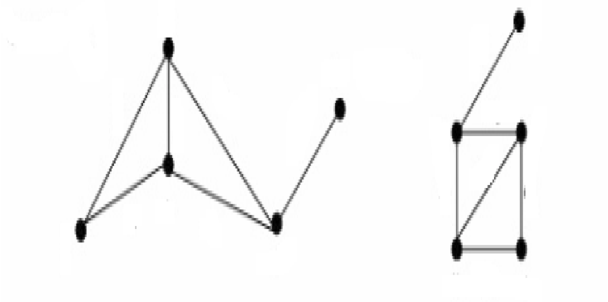
Graph $G(V, E)$ and $G^*(V^*, E^*)$ are said to be *isomorphic* if there exists a one-to-one correspondence $f : V \to V^*$ such that $\{u, v\}$ is an edge of G if and only if $\{f(u), f(v)\}$ is an edge of G^*.

Homeomorphic Graphs

Given any graph G, we can obtain a new graph by dividing an edge of G with additional vertices. Two graphs G and G^* are said to *homeomorphic* if they can be obtained from the same graph or isomorphic graphs by this method.

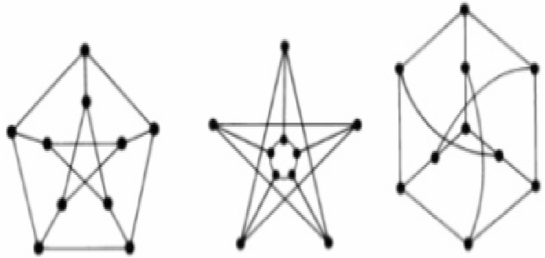
Question 7

Check whether the following two graphs are isomorphic or not.



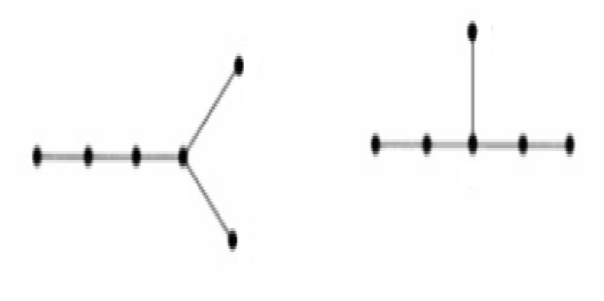
Question 8

Check whether the following three graphs are isomorphic or not.



Question 9

Check whether the following two graphs are isomorphic or not.



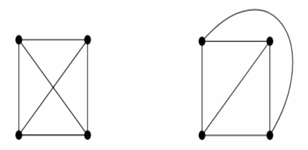
Definitions cont..

Planar Graph

A graph or multigraph which can be drawn in the plane so that its edges do not cross is said to be *planar*. When a planar graph is drawn without edges crossing, the edges and vertices of the graph divide the plane into regions. We will call each region a *face*.

Euler's Formula for planar graphs

For any (connected) planar graph with v vertices, e edges and f faces, we have $v - e + f = 2$.

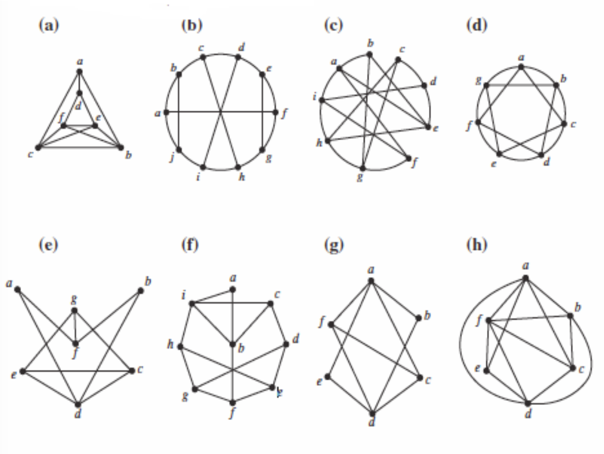


Kuratowski Theorem

A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

Question 10

Which of the following graphs are planar? $K_{3,3}$ or K_5 configurations in the nonplanar graphs.



Edge List

One simple way to represent a graph is just a list, or array, of E edges, which we call an *edge list*.

Adjacency List

The *Adjacency List* representation of an n-vertex graph consists of n lists, one list for each vertex i, $1 \geq i \geq n$, which records the vertices that i is adjacent to.

Adjacency Matrix

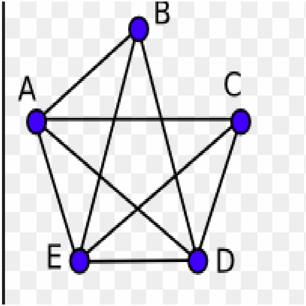
The *Adjacency Matrix* of a simple graph G whose rows and columns are indexed by the vertices. The entry (i, j) of this matrix is 1 if there is an edge from vertex i to vertex j and 0 otherwise.

Incidence Matrix

The incidence matrix of a graph with n vertices and m edges is an $n \times m$ 0-1 matrix with entry $[v, e] = 1$ if and only if vertex v is incident an edge e.

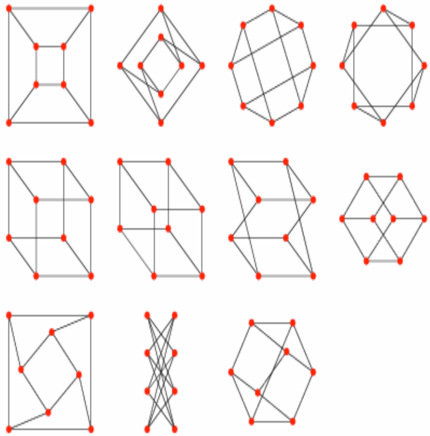
Question 11

Write down the Edge List, Adjacency List, Adjacency Matrix and Incidence Matrix of the following graph. Redo the question using **Maple**.



Graph Embedding

A Graph embedding is a particular drawing of a graph. Graph embedding provides an effective yet efficient way to solve the graph analytic problems.



Graph Embedding

There are several ways of embedding graph such as circular embedding, ranked embedding, radial embedding, rooted embedding, spring embedding etc.

Circular embedding

A circular embedding is a graph embedding in which all graph vertices lie on a common circle, usually arranged so they are equally spaced around the circumference.

Ranked embedding

In a ranked embedding, vertices are placed on evenly spaced vertical lines.

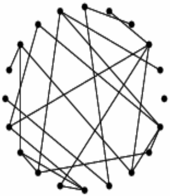
Rooted embedding

Rooted embedding is a way to embed graphs that represent hierarchies. one vertex is selected as a special vertex, a root, while the remaining vertices are ranked by the distance from the root.

Spring embedding

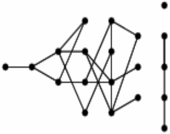
The graph is viewed as a system of bodies with forces acting on them.

Examples of some graph embeddings



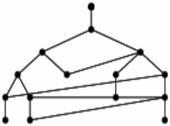
circular embedding

radial embedding



ranked embedding

spring embedding



rooted embedding