Complex Variables

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Lecture 10

Outline

- Let's analyze these zeroes of previous example $g(z) := z^5 + z + 1$ a little more.
- We keep the same function $g(z) := z^5 + z + 1$ but now take the reference function f to be the constant function $f(z) := 1$.
- And use the curve $\gamma'(t) = \frac{1}{2}e^{it} : 0 \leq t \leq 2\pi$, which traverses the circle $\left\{z \in C : |z| = \frac{1}{2}\right\}$.

- On this curve $|f(z)| = 1$, while $|g(z) - f(z)| = |z^5 + z| \leq |z^5| + |z| = \frac{1}{2^5} + \frac{1}{2} = \frac{17}{32}$
- Thus the condition of Rouche's theorem are again satisfied.
- And we see that g has the same number of zeroes inside γ' as f does, which is 0 (as 1 is never equal to 0).

- Thus g has no zeroes in the disk $z : |z| < 1/2$, which when combined with our previous analysis shows that all five zeroes must lie within the annulus $\{z : 1/2 < |z| < 2\}$.
- Note that g cannot have a zero on either γ or γ' by the argument contained in the proof of Rouche's theorem.
- In contrast with the behaviour on large circles, what is happening on small circles such as $\{z : |z| = 1/2\}$ is that constant term 1 of $z^5 + z + 1$ is dominating and so g has the same number of zeroes as 1 (as opposed to z^5, which is what happens for very large circles).

- One might hope to continue this analysis further and pinpoint exactly where the zeroes of f lie.
- Unfortunately for intermediate circles ($e.g.\ \{z : |z| = 1\}$) no single term in g is dominant.
- and it is not obvious how to use Rouche's theorem to count how many zeroes lie inside the unit circle.
- Like many tools in complex analysis Rouche's theorem is not a magic wand, but can give a lot of useful information nevertheless.

Outline

One consequence of Rouche's theorem is that it gives elegant proof of Fundamental Theorem of Algebra.

Theorem (Fundamental Theorem of Algebra)

Every polynomial $P(z)$ of degree n has exactly n zeroes (counting multiplicity), and thus can be factored completely into linear factors.

Proof

Write the polynomial as

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$$

and note that a_n is non-zero (otherwise P would have degree less than n).

Now choose a large circle $\{z : |z| = R\}$ for some $R >> 1$, on this circle we compare P against function $f(z) := a_n z^n$.

Note that

$$|f(z)| = |a_n z^n| = |a_n| R^n$$

and

$$|P(z) - f(z)| = |a_{n-1} z^{n-1} + ... + a_0| \leq |a_{n-1}| R^{n-1} + ... + |a_0|.$$

thus

$$\frac{|P(z) - f(z)|}{|f(z)|} \leq \frac{|a_{n-1}|}{|a_n| R} + \frac{|a_{n-2}|}{|a_n| R^2} + ... + \frac{|a_0|}{|a_n| R^n}$$

(recall that $|a_n|$ is non-zero). As $R \to +\infty$, everything on the right hand side tends to zero.

Thus there exists some R_0 such that for every $R > R_0$. We have

$$\frac{|a_{n-1}|}{|a_n|R} + \frac{|a_{n-2}|}{|a_n|R^2} + ... + \frac{|a_0|}{|a_n|R^n} < 1$$

and hence

$$|P(z) - f(z)| < |f(z)|$$

Thus by Rouche's theorem, P and f have the same number of zeroes in the circle $\{z \in C : |z| = R\}$. But f clearly has n zeroes in this circle. and thus P must also. Letting R go to infinity we obtain the result.

All the best for your exams!