

# Graph Theory and Its Applications

Dr. G.H.J. Lanel

Lecture 1

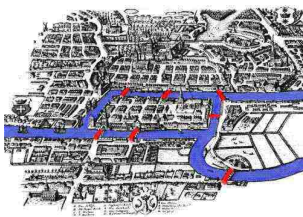
# Outline

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- 1 What is Graph Theory?
  - History
  - At present
- 2 Introduction to Graph Theory
  - Graphs
  - Properties of graphs
  - Special graphs
  - Bipartite graphs
  - Cycles
  - Directed graphs

# Seven Bridges of Konigsberg

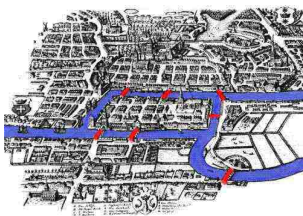
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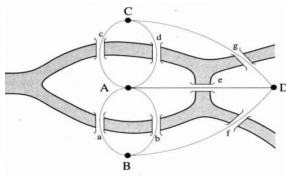
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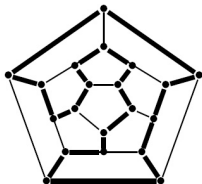




- In 1859 Sir William Rowan Hamilton developed a game that he sold to a Dublin toy manufacturer.
- The game consisted of a wooden regular dodecahedron with the 20 corner points labeled with the names of prominent cities.
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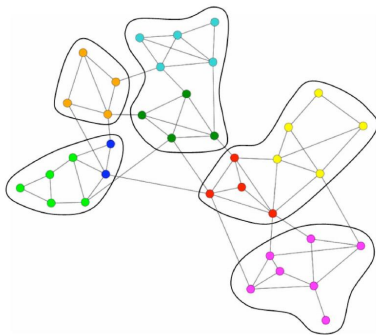


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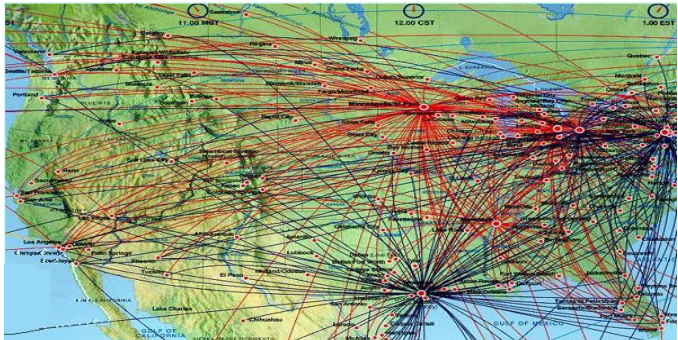
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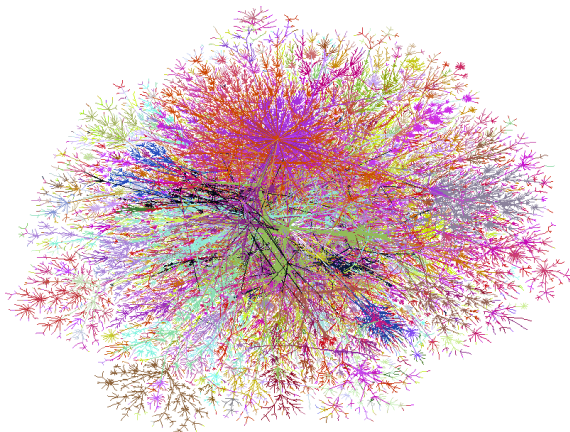


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# Ordering

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The screenshot shows a Google search interface with the search term 'university belgium'. The results are ordered by relevance, with the most relevant results appearing first. The top result is 'Portaal Universiteit Gent/Ghent University Web portal', followed by 'Université de Liège - University of Liège (Belgium)', and 'Service Télématique et Communication'. Each result includes a brief description and a category path.

Recherche avancée Préférences Outils linguistiques Conseils de recherche

university belgium Recherche Google

Rechercher dans : Web Pages francophones Pages : France

Web Images Groupes Répertoire

Google a recherché university belgium sur le Web. 1 - 10 résultats, sur un total d'environ 1,040,000. Recherche effectuée

Voulez-vous limiter la recherche à la langue : Français ?

Catégorie: [Regional > Europe > ... > Education > Non-University Higher Education](#)

[Portaal Universiteit Gent/Ghent University Web portal](#)  
 ... U bent NIET ingelogd. Log in. UNIVERSITEIT GENT - Nederlandstalige site.  
 GHENT UNIVERSITY - English site. ©2002 Universiteit Gent, Disclaimer.  
 Description: The largest and oldest public university in Belgium. Site in both Dutch and English. Links to education, ...  
 Catégorie: [Reference > Education > ... > Europe > Belgium > Ghent University](#)  
[www.rug.ac.be/ - 7k - En cache - Pages similaires](#)

[Université de Liège - University of Liège \(Belgium\)](#)  
 L'Université de Liège, une Université complète : 8 facultés, 32 filières d'enseignement, 350 unités de recherche  
 Description: ULG - Présentation de l'institution, la recherche et de l'enseignement. Guide du futur étudiant.  
 Catégorie: [World > Français > ... > Belgique > Université de Liège](#)  
[www.ulg.ac.be/ - 3k - En cache - Pages similaires](#)

[Service Télématique et Communication](#)  
 IHE. Main Areas. People. Internal Reports. Newsletter. Books. Summer  
 fête 2002. Wireless seminar. Webmaster. STC works with or is involved ...  
[www.ihe.ac.be/ - 13k - En cache - Pages similaires](#)

[Universiteit Antwerpen](#)  
 Welkom aan de Universiteit Antwerpen, ...  
 Description: De studies, voorzieningen, onderwijs, onderzoek en nieuws.  
 Catégorie: [World > Nederlands > ... > Gemeenten > Antwerpen > Onderwijs](#)  
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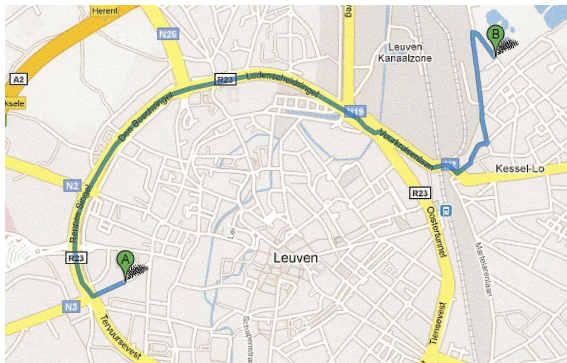
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# Definition of a graph

- Perhaps the most useful object in discrete mathematics (especially for computer science and other applications) is a structure called a graph.
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For example,

Here  $V = \{v_1, v_2, \dots, v_5\}$  and  $E = \{e_1, e_2, \dots, e_6\}$ . An edge  $e_k = (v_i, v_j)$  is incident with the vertices  $v_i$  and  $v_j$ .

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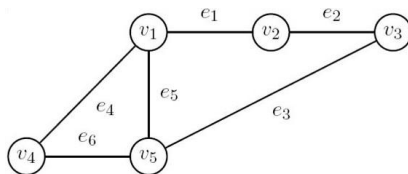
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- A **self-loop** is an edge that joins to an identical vertex.
- A **multi-edge** is a collection of two or more edges having distinct end vertices.

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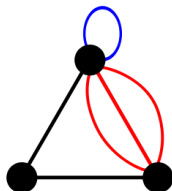
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- The **degree**  $d(v)$  of a vertex  $v \in V$  is the number of edges incident to  $v$ .

- **Proposition:** Let  $G = (V, E)$  be a graph. Then

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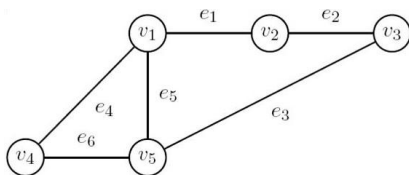
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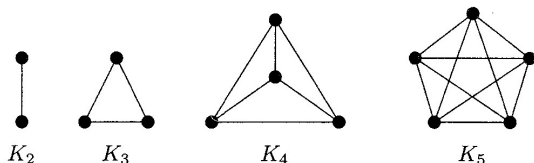
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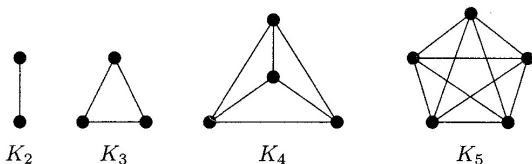
- A **complete** graph  $K_n$  is a simple graph with  $n(n-1)/2$  possible edges. When  $n = 2, 3, 4, 5$ , we have the following graphs.



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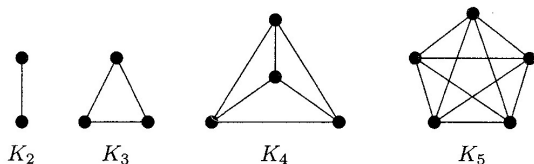


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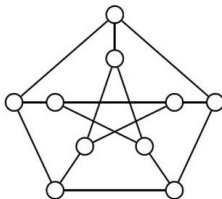
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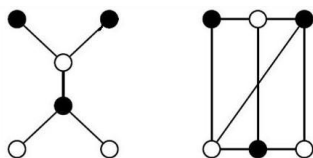


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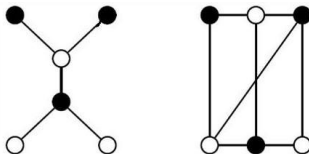
- A **bipartite** graph is one where  $V = V_1 \cup V_2$  such that there are edges only between  $V_1$  and  $V_2$  (the black and white nodes below).



- A **complete bipartite** graph is one where all edges between  $V_1$  and  $V_2$  are present (i.e.  $E = |V_1| \cdot |V_2|$ ). It is noted as  $K_{n_1, n_2}$ , where  $n_1 = |V_1|$  and  $n_2 = |V_2|$ .

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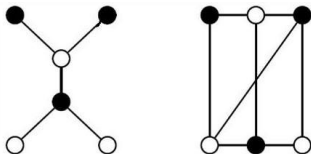
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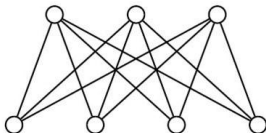
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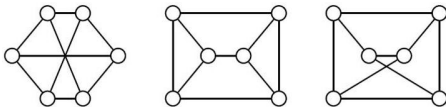


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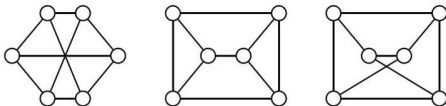
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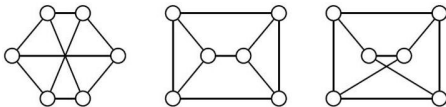
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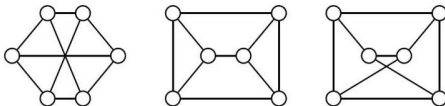
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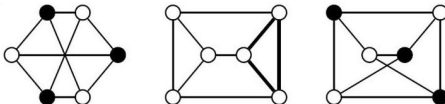
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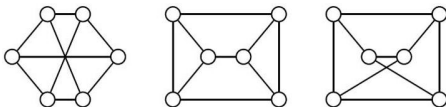
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- A **walk** of length  $k$  from vertex  $v_0$  to vertex  $v_k$  is a non-empty graph  $W = (V_1, E_1)$  of the form

$$V_1 = \{v_0, v_1, \dots, v_k\}, E_1 = \{(v_0, v_1), \dots, (v_{k-1}, v_k)\},$$

where edge  $j$  connects vertices  $j - 1$  and  $j$  (i.e.  $|V_1| = |E_1| + 1$ ).

- A **trail** is a walk with all different edges.
- A walk or trail is closed when  $v_0 = v_k$ .
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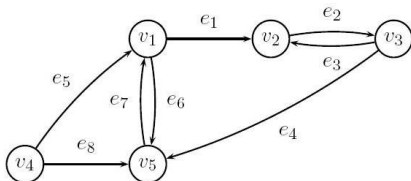
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