Graph Theory and Its Applications

Dr. G.H.J. Lanel

Lecture 1

Outline

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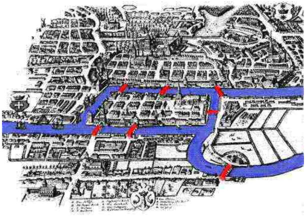
Seven Bridges of Konigsberg

Graph theory started with Euler who was asked to find a nice path across the seven Koningsberg bridges.



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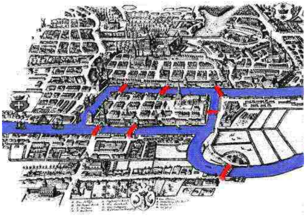

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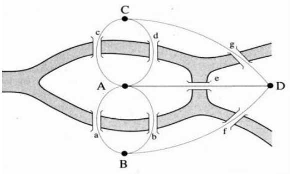

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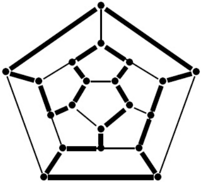

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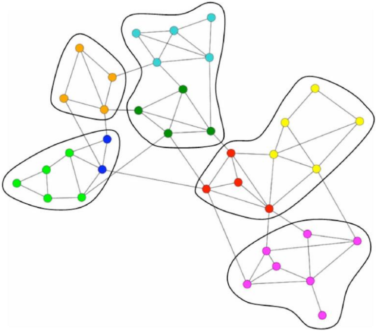

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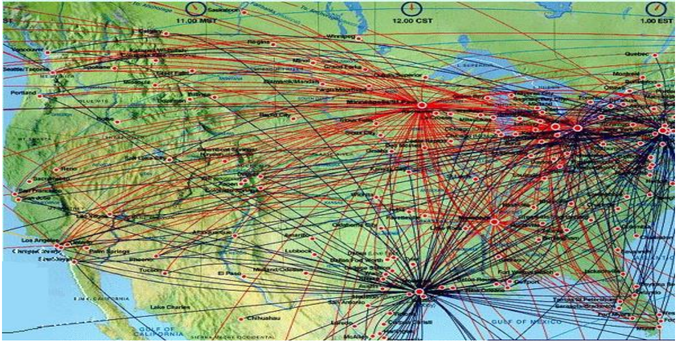

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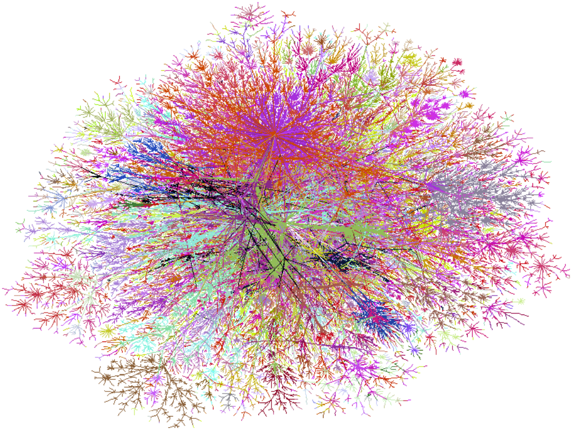

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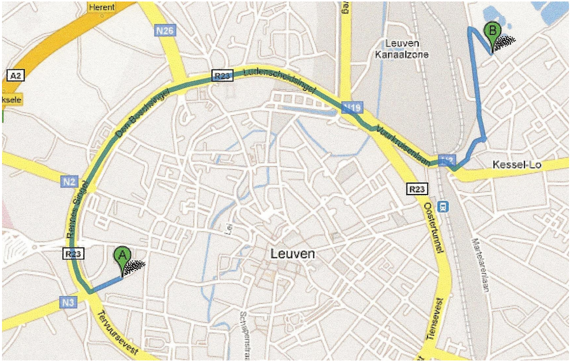

Outline

1. What is Graph Theory?
   - History
   - At present

2. Introduction to Graph Theory
   - Graphs
   - Propoties of graphs
   - Special graphs
   - Bipartite graphs
   - Cycles
   - Directed graphs

Definition of a graph

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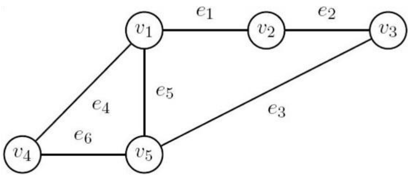

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Simple graphs

- A self-loop is an edge that joins to an identical vertex.
- A multi-edge is a collection of two or more edges having distinct end vertices.

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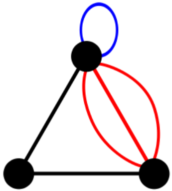

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Simple graphs

- The degree $d(v)$ of a vertex $v \in V$ is the number of edges incident to v.
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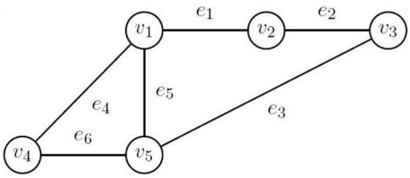

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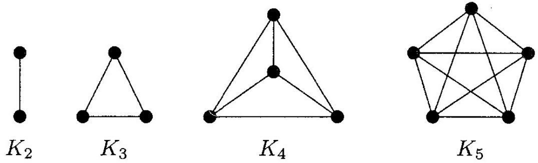

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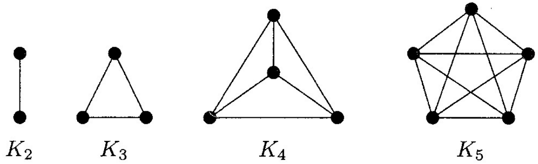

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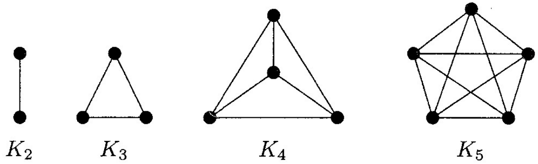

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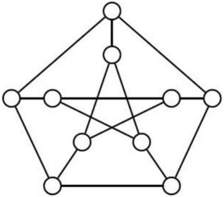

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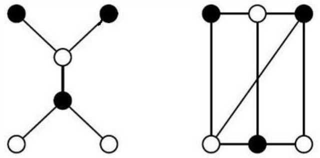

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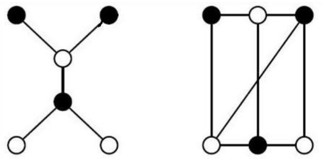

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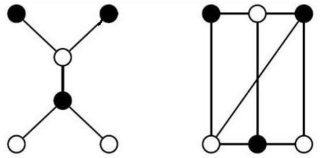

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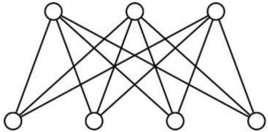

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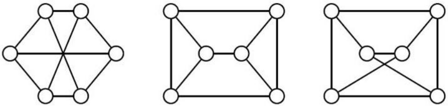

- It suffices to find two colors that separate the edges as below,

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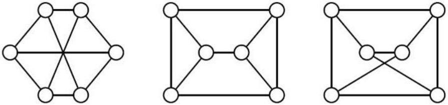

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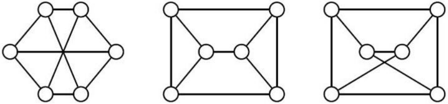

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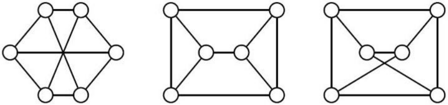

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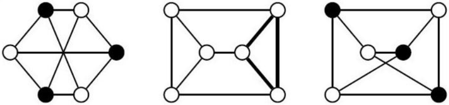

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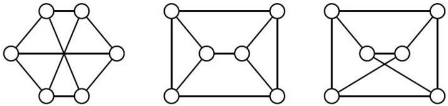

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- A walk of length k from vertex v_0 to vertex v_k is a non-empty graph $W = (V_1, E_1)$ of the form

$$V_1 = \{v_0, v_1 \cdots, v_k\}, E_1 = \{(v_0, v_1), \cdots, (v_{k-1}, v_k)\},$$

where edge j connects vertices $j - 1$ and j (i.e. $|V_1| = |E_1| + 1$).

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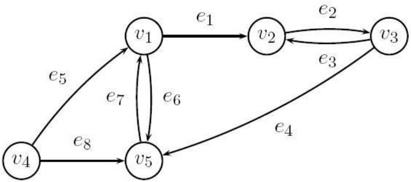

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