MAT 122 2.0 Calculus

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Lecture 1

Outline

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- A **sequence** is an ordered collection of objects in which repetitions are allowed. For example,

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

- we can obtain a **subsequence** by restricting the original sequence to a smaller index set. For example,

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The n^{th} term

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How could you write the n^{th} term of the following sequences?

1. $1 + \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
2. $1, 2, 4, 8, 16, \ldots$
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Answers

1. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \Rightarrow a_n = \frac{1}{n}$
2. $1, 2, 4, 8, 16, ..., \Rightarrow a_n = 2^{n-1}$
3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ..., \Rightarrow a_n = \frac{n}{n+1}$

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Monotonic sequence

- A sequence is called **increasing**, if $a_n < a_{n+1}$ for all $n \geq 1$. That is $a_1 < a_2 < a_3...$ for all n.
- It is called **decreasing**, if $a_n > a_{n+1}$ for all $n \geq 1$.
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Consider the following examples:

The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing since

$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$, for all $n \geq 1$.

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Show that the sequence $\left\{\frac{n}{n^2+1}\right\}$ is decreasing.

Solution:

We must show that $a_{n+1} < a_n$, that is, $\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$

$$\begin{aligned}\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} &\Leftrightarrow (n+1)(n^2+1) < n[(n+1)^2+1] \\ &\Leftrightarrow n^3+n^2+n+1 < n^3+2n^2+2n \\ &\Leftrightarrow 1 < n^2+n\end{aligned}$$

Since $n \geq 1$, $n^2+1 > 1$ we know that the inequality is true. Therefore, $a_{n+1} < a_n$ and so sequence is decreasing.

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Bounded sequence

- A sequence $\{a_n\}$ is **bounded above** if there is a number M such that,

$$a_n \leq M, \text{ for all } n \geq 1.$$

- It is **bounded below** if there is a number m such that,

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Examples:

- The sequence $\{n\}$ is bounded below by 0 but not above.
- The sequence $\left\{\frac{n}{n+1}\right\}$ is bounded below and above by 0 and 1 respectively for all n.

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A sequence $\{a_n\}$ has the limit L and we write

$\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$.

i.e. if for every $\epsilon > 0$ there is a corresponding integer N such that $a_n - L < \epsilon$, whenever $n > N$.

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If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is convergent). Otherwise, we say the sequence **diverges** (or is divergent).

Examples:

Convergent sequences have a finite limit.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \Rightarrow a_n = \frac{1}{n}$ and the limit of the sequence is 0

Divergent sequences do not have a finite limit.

$3, 5, 7, 9, 11, 13, ..., \Rightarrow a_n = 2n + 1$ and the limit of the sequence is undefined.

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Limit Laws

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- $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} c * a_n = c * \lim_{n\to\infty} a_n$, for some real number c
- $\lim_{n\to\infty} a_n * \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n * b_n$
- If $\lim_{n\to\infty} b_n \neq 0$, then $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$
- If $p > 0$ and $a_n > 0$, then $\lim_{n\to\infty} a_n^p = (\lim_{n\to\infty} a_n)^p$

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2. $\lim_{n\to\infty} (-1)^n/n$
3. $\lim_{n\to\infty} (1+1/n)^n$
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