Graph Theory and Its Applications

Dr. G.H.J. Lanel

Lecture 2

Outline

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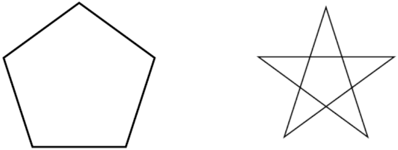
- Two graphs are isomorphic if there is a 1-1 correspondence between their vertex sets that preserves adjacencies and non-adjacencies.



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- This problem is still believed to be NP hard.

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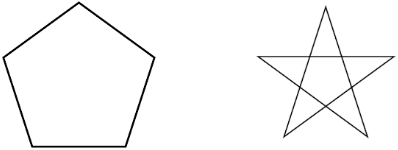

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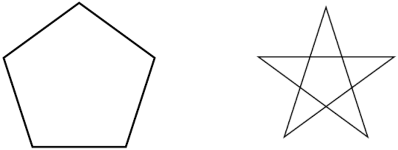

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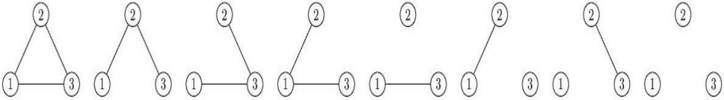
- How many different simple graphs are there with n vertices?
- A graph with n vertices can have at most $n(n-1)/2$ different edges and each of them can be present or not.



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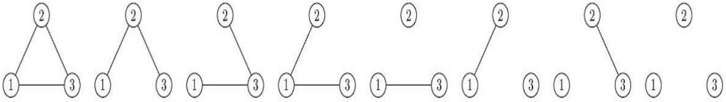

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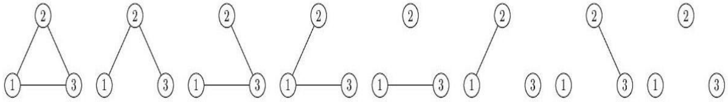

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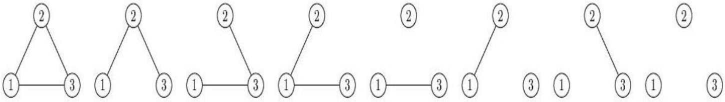

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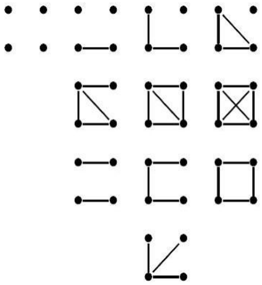
- With $n = 4$ one finds eventually 11 different graphs after collapsing the isomorphic ones.



- Let there be T_n non-isomorphic graphs with n vertices. Then

$$\frac{2^{n(n-1)/2}}{n!} \leq T_n \leq 2^{n(n-1)/2}$$

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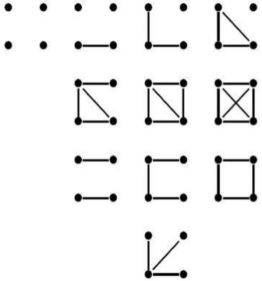

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The complement of $G = (V, E)$, denoted $\bar{G}$, is the graph $\bar{G} = (V, E^{'})$ with same vertex set as G whose edges are precisely the edges missing from G.

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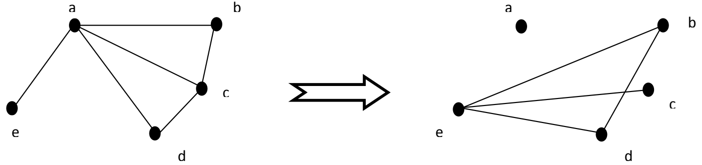

The line graph $L(G)$ of a graph G has a vertex for each edge of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common.

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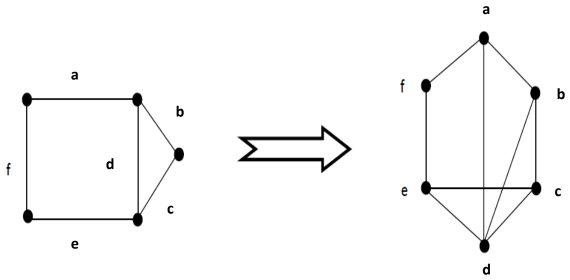

- An eulerian trail in a graph is a trail that contains every edge of that graph.
- An eulerian tour is a closed eulerian trail.
- An eulerian graph is a graph that has an eulerian trail.

Eg:-

The trail $T = \{a, e, b, a, f, b, c, f, g, d, c, a\}$ is an eulerian tour.

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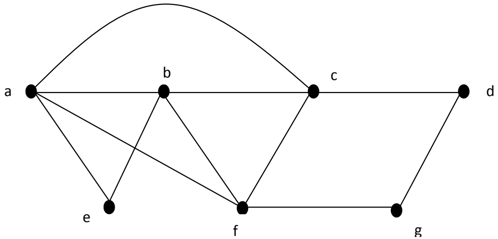

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Theorem

A connected graph G with at least one edge has an euler tour if and only if the degree of every vertex in G is even.

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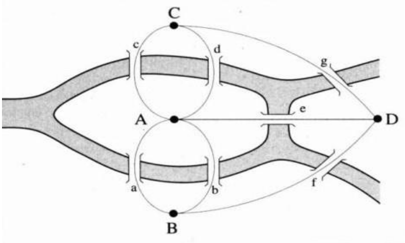

- An cycle that includes every vertex of a graph is called a Hamiltonian cycle.
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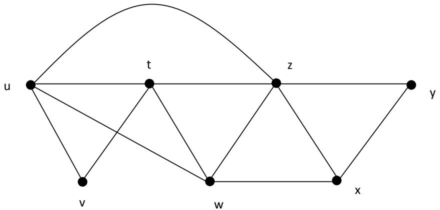

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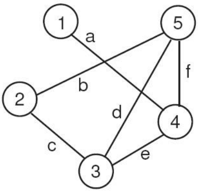
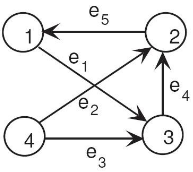
Adjacency matrix

A graph $G = (V, E)$ is often represented by its adjacency matrix. It is an $n \times n$ matrix A with $A(i, j) = 1$ iff $(i, j) \in E$. Examples:





The adjacency matrices are

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

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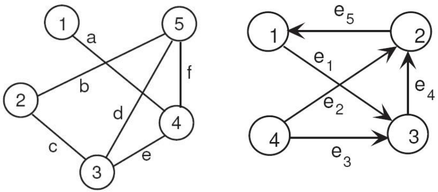

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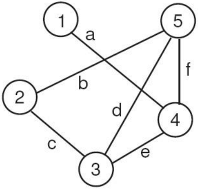

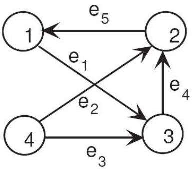

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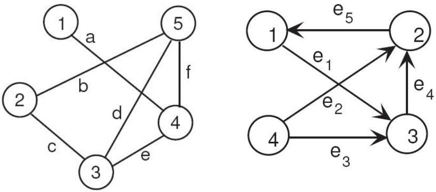
Incidence matrix

A graph can also be represented by its $n \times m$ incidence matrix T. For an undirected graph $T(i,k) = T(j,k) = 1$ *iff* $e_k = (v_i, v_j)$. For a directed graph $T(i,k) = -1, T(j,k) = 1$ *iff* $e_k = (v_i, v_j)$. Examples:



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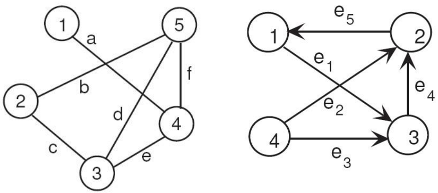

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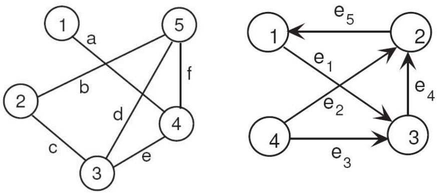

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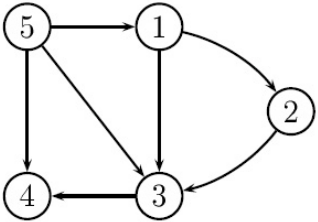
One can also use a sparse matrix representation of A and T. This is in fact nothing but a list of edges, organized by vertices.



$$V(1) = \{2, 3\}$$
$$V(2) = \{3\}$$
$$V(3) = \{4\}$$
$$V(4) = \emptyset$$
$$V(5) = \{1, 3, 4\}$$

End!