Limits and Continuity

Dr. G.H.J. Lanel

Lecture 3

Outline

Outline

Limit of a Function

Intuitive Definition

Suppose f is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.)

Then write,

$$\lim_{x \to a} f(x) = L$$

Limit of a Function

Intuitive Definition

Suppose f is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.)

Then write,

$$\lim_{x \to a} f(x) = L$$

Limit of a Function

Intuitive Definition

Suppose f is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.)

Then write,

$$\lim_{x \to a} f(x) = L$$

And say

"the limit of f, as x approaches a, equals L?

If we can make the values of f arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

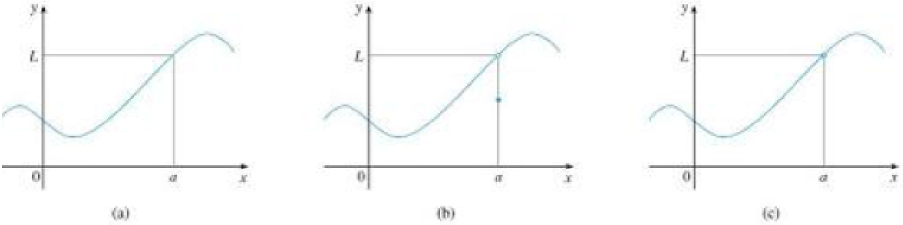
And say

”the limit of f, as x approaches a, equals L?

If we can make the values of f arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Exercise 1

What is the limit as x approaches a in each graph?



Exercise 2

Find the limit numerically:

$$\lim_{x \to 1} \frac{x-1}{x^2-1}$$

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

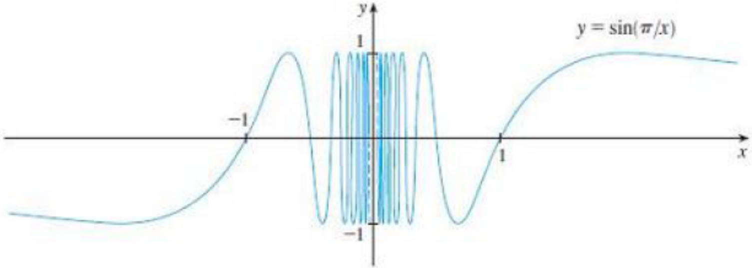
$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

Exercise 3

Find the limit numerically :

$$\lim_{x \to 0} sin(\frac{\pi}{x})$$

(Try $x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{10}, \frac{1}{100}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \frac{2}{11}, \frac{2}{101}$)



One Sided Limits

Intuitive Definition

$$\lim_{x \to a^-} f(x) = L$$

and say the left-hand limit of f as x approaches a [or the limit of f as x approaches a from the left] is equal to L if we can make the values of f arbitrarily close to L by taking x to be sufficiently close to a with x less than a.

One Sided Limits

Intuitive Definition

$$\lim_{x \to a^-} f(x) = L$$

and say the left-hand limit of *f* as *x* approaches *a* [or the limit of *f* as *x* approaches *a* from the left] is equal to *L* if we can make the values of *f* arbitrarily close to *L* by taking *x* to be sufficiently close to *a* with *x* less than *a*.

One Sided Limits

Intuitive Definition

$$\lim_{x \to a^+} f(x) = L$$

and say the right-hand limit of f as x approaches a [or the limit of f as x approaches a from the right] is equal to L if we can make the values of f arbitrarily close to L by taking x to be sufficiently close to a with x greater than a.

One Sided Limits

Intuitive Definition

$$\lim_{x \to a^+} f(x) = L$$

and say the right-hand limit of f as x approaches a [or the limit of f as x approaches a from the right] is equal to L if we can make the values of f arbitrarily close to L by taking x to be sufficiently close to a with x greater than a.

Theorem

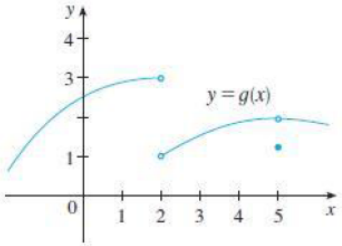
$$\lim_{x \to a} f(x) = L$$

$$iff$$

$$\lim_{x \to a^+} f(x) = L \; and \; \lim_{x \to a^-} f(x) = L$$

Exercise 4

Find each limit (*if* it exists.)

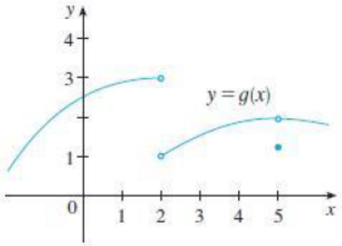


$$(a)\ \lim_{x\to 2^-} g(x),\ \lim_{x\to 2^+} g(x),\ \lim_{x\to 2} g(x)$$

$$(b)\ \lim_{x\to 5^-} g(x),\ \lim_{x\to 5^+} g(x),\ \lim_{x\to 5} g(x)$$

Exercise 4

Find each limit (*if* it exists.)



$$(a)\ \lim_{x \to 2^-} g(x),\ \lim_{x \to 2^+} g(x),\ \lim_{x \to 2} g(x)$$

$$(b)\ \lim_{x \to 5^-} g(x),\ \lim_{x \to 5^+} g(x),\ \lim_{x \to 5} g(x)$$

.

Exercise 5

Find the limit

$$\lim_{x \to 0} \frac{|x|}{x}$$

Infinite Limits

Intuitive Definition

Let f be a function defined on both sides of a, except possibly at a itself. Then,

$$\lim_{x \to a} f(x) = \infty$$

means that the values of f can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Infinite Limits

Intuitive Definition

Let f be a function defined on both sides of a, except possibly at a itself. Then,

$$\lim_{x \to a} f(x) = \infty$$

means that the values of f can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Infinite Limits

Intuitive Definition

Let f be a function defined on both sides of a, except possibly at a itself. Then,

$$\lim_{x \to a} f(x) = \infty$$

means that the values of f can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Intuitive Definition

$$\lim_{x \to a} f(x) = -\infty$$

Means that the values of f can be made arbitrarily large negatively by taking x sufficiently close to a, but not equal to a.

Exercise 6

Find

$$(a) \lim_{x \to 0} \frac{1}{x^2}$$

$$(b) \lim_{x \to 0} \frac{-1}{x^2}$$

Exercise 6

Find

$$(a) \lim_{x \to 0} \frac{1}{x^2}$$

$$(b) \lim_{x \to 0} \frac{-1}{x^2}$$

Exercise 7

Find

$$(a) \lim_{x \to 0^-} \frac{1}{x}$$

$$(b) \lim_{x \to 0^+} \frac{1}{x}$$

$$(c) \lim_{x \to 0} \frac{1}{x}$$

Exercise 7

Find

$$(a) \lim_{x \to 0^-} \frac{1}{x}$$

$$(b) \lim_{x \to 0^+} \frac{1}{x}$$

$$(c) \lim_{x \to 0} \frac{1}{x}$$

Exercise 7

Find

$$(a) \lim_{x\to 0^-} \frac{1}{x}$$

$$(b) \lim_{x\to 0^+} \frac{1}{x}$$

$$(c) \lim_{x\to 0} \frac{1}{x}$$

Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x)$$

and

$$\lim_{x \to a} g(x)$$

exist. Then,

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$\lim_{x \to a} [f(x).g(x)] = \lim_{x \to a} f(x). \lim_{x \to a} g(x)$$

Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x)$$

and

$$\lim_{x \to a} g(x)$$

exist. Then,

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$\lim_{x \to a} [f(x).g(x)] = \lim_{x \to a} f(x). \lim_{x \to a} g(x)$$

Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x)$$

and

$$\lim_{x \to a} g(x)$$

exist. Then,

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$\lim_{x \to a} [f(x).g(x)] = \lim_{x \to a} f(x). \lim_{x \to a} g(x)$$

Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x)$$

and

$$\lim_{x\to a} g(x)$$

exist. Then,

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x). \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)},\ if\ \lim_{x\to a}g(x)\neq 0$$

$$\lim_{x\to a}f(x)^n=\left(\lim_{x\to a}f(x)\right)^n,\ n\in\mathbb{Z}$$

$$\lim_{x\to a}cf(x)=c\lim_{x\to a}f(x)$$

$$\lim_{x\to a}c=c$$

$$\lim_{x\to a}x=a$$

$$\lim_{x\to a}x^n=a^n$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \textit{if } \lim_{x\to a} g(x) \neq 0$$

$$\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n, n \in \mathbb{Z}$$

$$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$$

$$\lim_{x\to a} c = c$$

$$\lim_{x\to a} x = a$$

$$\lim_{x\to a} x^n = a^n$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}, \textit{if } \lim_{x\to a}g(x)\neq 0$$

$$\lim_{x\to a}f(x)^n=\left(\lim_{x\to a}f(x)\right)^n, n\in\mathbb{Z}$$

$$\lim_{x\to a}cf(x)=c\lim_{x\to a}f(x)$$

$$\lim_{x\to a}c=c$$

$$\lim_{x\to a}x=a$$

$$\lim_{x\to a}x^n=a^n$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \textit{if } \lim_{x \to a} g(x) \neq 0$$

$$\lim_{x \to a} f(x)^n = \left(\lim_{x \to a} f(x)\right)^n, n \in \mathbb{Z}$$

$$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$$

$$\lim_{x \to a} c = c$$

$$\lim_{x \to a} x = a$$

$$\lim_{x \to a} x^n = a^n$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}, \textit{if} \lim_{x\to a}g(x)\neq 0$$

$$\lim_{x\to a}f(x)^n=\left(\lim_{x\to a}f(x)\right)^n, n\in\mathbb{Z}$$

$$\lim_{x\to a}cf(x)=c\lim_{x\to a}f(x)$$

$$\lim_{x\to a}c=c$$

$$\lim_{x\to a}x=a$$

$$\lim_{x\to a}x^n=a^n$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}, \textit{if } \lim_{x\to a}g(x)\neq 0$$

$$\lim_{x\to a}f(x)^n=\left(\lim_{x\to a}f(x)\right)^n, n\in\mathbb{Z}$$

$$\lim_{x\to a}cf(x)=c\lim_{x\to a}f(x)$$

$$\lim_{x\to a}c=c$$

$$\lim_{x\to a}x=a$$

$$\lim_{x\to a}x^n=a^n$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \textit{if} \lim_{x \to a} g(x) \neq 0$$

$$\lim_{x \to a} f(x)^n = \left(\lim_{x \to a} f(x)\right)^n, n \in \mathbb{Z}$$

$$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$$

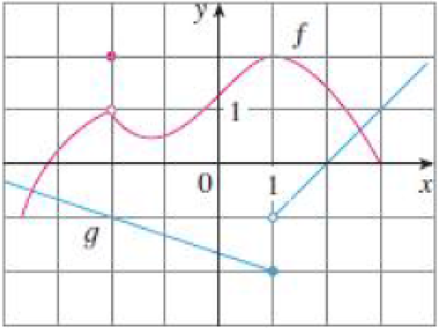
$$\lim_{x \to a} c = c$$

$$\lim_{x \to a} x = a$$

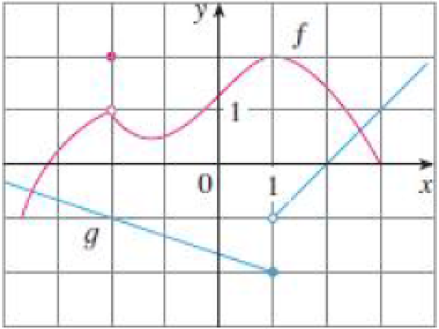
$$\lim_{x \to a} x^n = a^n$$

Exercise 8

Use the Limit Laws and the graphs of f and g in Figure to evaluate the following limits, if they exist.



$$(i) \lim_{x \to -2} (f(x) + 3g(x))$$



$$(ii)\ \lim_{x \to 1} f(x).g(x)$$

$$(iii)\ \lim_{x \to 2} \frac{f(x)}{g(x)}$$

Direct Substitution Property

If f is a polynomial or a rational function and a is in the domain of f then,

$$\lim_{x \to a} f(x) = f(a)$$

Exercise 9

Find

$$\lim_{x \to 2} \frac{\sqrt{1 + x^2}}{x - 3}$$

The Squeeze Theorem

The Squeeze Theorem

If f(x)$\leq$ g(x) $\leq$ h(x) when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Exercise 10

Find

$$\lim_{x \to 0} x^2 sin\frac{1}{x}$$

Outline

Continuity

Definition

A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

f is discontinuous at a (or f has a discontinuity at a) if f is not continuous at a.

A discontinuity may be a removable discontinuity, jump discontinuity or an infinite discontinuity.

Continuity

Definition

A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

f is discontinuous at a (or f has a discontinuity at a) if f is not continuous at a.

A discontinuity may be a removable discontinuity, jump discontinuity or an infinite discontinuity.

Continuity

Definition

A function f is continuous at a number a if

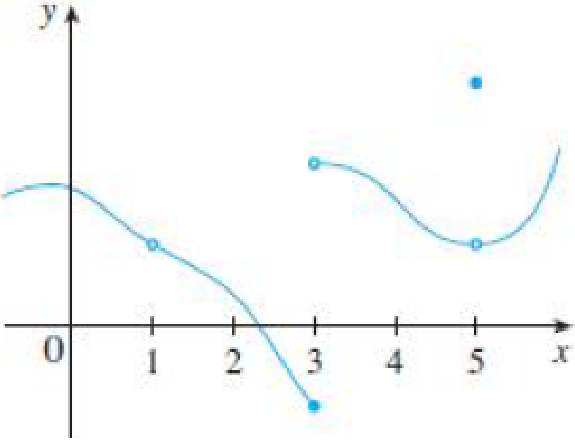
$$\lim_{x\to a} f(x) = f(a)$$

f is discontinuous at a (or f has a discontinuity at a) if f is not continuous at a.

A discontinuity may be a removable discontinuity, jump discontinuity or an infinite discontinuity.

Exercise 11

At which numbers is f discontinuous? Why?



Exercise 12

Where are each of the following functions discontinuous?

$$f(x) = \frac{1 - x^2}{1 - x}$$

$$g(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{Otherwise} \end{cases}$$

Exercise 12

Where are each of the following functions discontinuous?

$$f(x) = \frac{1 - x^2}{1 - x}$$

$$g(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{Otherwise} \end{cases}$$

- A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

- A function f is continuous from the left at a number a if

$$\lim_{x \to a^-} f(x) = f(a)$$

- A function f is continuous on an interval if it is continuous at every number in the interval.

- A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

- A function f is continuous from the left at a number a if

$$\lim_{x \to a^-} f(x) = f(a)$$

- A function f is continuous on an interval if it is continuous at every number in the interval.

- A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

- A function f is continuous from the left at a number a if

$$\lim_{x \to a^-} f(x) = f(a)$$

- A function f is continuous on an interval if it is continuous at every number in the interval.

- A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

- A function f is continuous from the left at a number a if

$$\lim_{x \to a^-} f(x) = f(a)$$

- A function f is continuous on an interval if it is continuous at every number in the interval.

- If f is defined only on one side of an endpoint of the interval,it is continuous at the endpoint to mean continuous from the right or continuous from the left.
- If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a:

  $f+g, f-g, fg, cf$, and f/g with $g(a) \neq 0$.
- The following types of functions are continuous at every number in their domains:
  polynomials, *rational functions, root functions, exponential functions, logarithmic functions, trigonometric functions*.

- If f is defined only on one side of an endpoint of the interval,it is continuous at the endpoint to mean continuous from the right or continuous from the left.
- If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a:

  $f+g, f-g, fg, cf$, and f/g with $g(a) \neq 0$.
- The following types of functions are continuous at every number in their domains:
  polynomials, *rational functions, root functions, exponential functions, logarithmic functions, trigonometric functions*.

- If f is defined only on one side of an endpoint of the interval,it is continuous at the endpoint to mean continuous from the right or continuous from the left.

- If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a:

  $f + g, f - g, fg, cf$, and f/g with $g(a) \neq 0$.

- The following types of functions are continuous at every number in their domains:
  polynomials, *rational functions, root functions, exponential functions, logarithmic functions, trigonometric functions*.

Exercise 13

On what intervals is each function continuous?

- $f(x) = x^{10} - \pi x^5 + 7$
- $g(x) = \sqrt{x} + \dfrac{1+x}{1-x}$

If f is continuous at b and

$$\lim_{x \to a} g(x) = b$$

then,

$$\lim_{x \to a} f[g(x)] = f(b)$$

Exercise 13

On what intervals is each function continuous?

- $f(x) = x^{10} - \pi x^5 + 7$
- $g(x) = \sqrt{x} + \dfrac{1+x}{1-x}$

If f is continuous at b and

$$\lim_{x \to a} g(x) = b$$

then,

$$\lim_{x \to a} f[g(x)] = f(b)$$

Exercise 13

On what intervals is each function continuous?

- $f(x) = x^{10} - \pi x^5 + 7$
- $g(x) = \sqrt{x} + \dfrac{1+x}{1-x}$

If f is continuous at b and

$$\lim_{x \to a} g(x) = b$$

then,

$$\lim_{x \to a} f[g(x)] = f(b)$$

Theorem

Theorem

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Exercise 14

On which interval

$$f(x) = \sin(x^2)$$

continuous?

Intermediate Value Theorem

Definition

Suppose that f is continuous on the closed interval $[a; b]$ and let N be any number between $f(a)$ sad and $f(b)$, where $f(a) \neq$f(b).

Then there exists a number $c \in (a, b)$ such that $f(c) = N$.

Exercise 15

Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

Outline

Derivatives

Definition

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(h)}{h}$$

provided that this limit exists.

- The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope $m = f'(a)$.

Derivatives

Definition

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(h)}{h}$$

provided that this limit exists.

- The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope $m = f'(a)$.

Exercise 16

Find an equation of the tangent line to the parabola y = x^2 at the point $P(1, 1)$.

Exercise 17

A manufacturer produces bolts of a fabric with a fixed width.

The cost of producing x meters of this fabric is $C = f(x)$ rupees.

- What is the meaning of the derivative $f'(x)$? What are its units?
- In practical terms, what does it mean to say that $f'(1000) = 9$?

Notations

- If replace a by x which is a variable, it regard $f'(x)$ as a function.

Notations for derivative of $y = f(x)$

$$y' = y'(x) = f' = f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Notations

- If replace a by x which is a variable, it regard $f'(x)$ as a function.

Notations for derivative of $y = f(x)$

$$y' = y'(x) = f' = f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Definition

Definition

A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (-∞ ,b) or a,∞) or (-∞,∞)] if it is differentiable at every number in the interval.

Exercise 18

On which interval f(x) = $|x|$ differentiable?

Theorem

Theorem

If f is differentiable at a, then f is continuous at a.

- A function may fail to be differentiable due to a discontinuity, a corner, or a vertical tangent.
- If f is a differentiable function, then its derivative f' is also a function.

  f' may have a derivative of its own, denoted by $(f')' = f''$.

  This new function f'' is called the *second derivative* of f because it is the derivative of the derivative of f.

Theorem

Theorem

If f is differentiable at a, then f is continuous at a.

- A function may fail to be differentiable due to a discontinuity, a corner, or a vertical tangent.
- If f is a differentiable function, then its derivative f' is also a function.

  f' may have a derivative of its own, denoted by $(f')' = f''$.

  This new function f'' is called the *second derivative* of f because it is the derivative of the derivative of f.

Theorem

Theorem

If f is differentiable at a, then f is continuous at a.

- A function may fail to be differentiable due to a discontinuity, a corner, or a vertical tangent.
- If f is a differentiable function, then its derivative f' is also a function.

  f' may have a derivative of its own, denoted by $(f')' = f''$.

  This new function f'' is called the *second derivative* of f because it is the derivative of the derivative of f.

- Notations for higher order derivatives; $f''(x)$, $\dfrac{d^2y}{dx^2}$, ... ,$f^n(x)$, $\dfrac{d^ny}{dx^n}$
- If $s = s(t)$ is the position function of an object that moves in a straight line, its first derivative represents the (instantaneous) velocity $v(t)$ of the object as a function of time.

  The instantaneous rate of change of velocity with respect to time is called the acceleration $a(t)$ of the object.

  Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function.

- Notations for higher order derivatives; $f''(x)$, $\dfrac{d^2y}{dx^2}$, ... ,$f^n(x)$, $\dfrac{d^ny}{dx^n}$
- If $s = s(t)$ is the position function of an object that moves in a straight line, its first derivative represents the (instantaneous) velocity $v(t)$ of the object as a function of time.

The instantaneous rate of change of velocity with respect to time is called the acceleration $a(t)$ of the object.

Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function.

- Notations for higher order derivatives; $f''(x)$, $\frac{d^2y}{dx^2}$, ... ,$f^n(x)$, $\frac{d^ny}{dx^n}$
- If $s = s(t)$ is the position function of an object that moves in a straight line, its first derivative represents the (instantaneous) velocity $v(t)$ of the object as a function of time.

  The instantaneous rate of change of velocity with respect to time is called the acceleration $a(t)$ of the object.

  Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function.

- Notations for higher order derivatives; $f''(x)$, $\dfrac{d^2y}{dx^2}$, ... ,$f^n(x)$, $\dfrac{d^ny}{dx^n}$

- If $s = s(t)$ is the position function of an object that moves in a straight line, its first derivative represents the (instantaneous) velocity $v(t)$ of the object as a function of time.

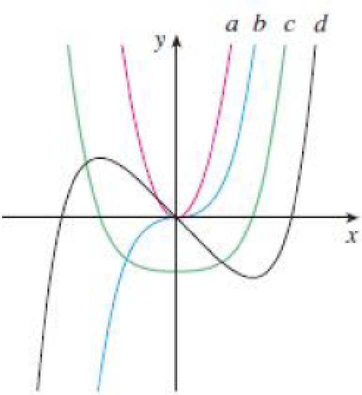
  The instantaneous rate of change of velocity with respect to time is called the acceleration $a(t)$ of the object.

  Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function.

Exercise 19

Figure shows graphs of f, f', f'', and f'''.

Identify each curve, and explain your choices.



Exercise 20

Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30*ml* of ethanol (corresponding to two alcoholic drinks).

t (hours)	1	1.5	2	2.5	3
$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007

- Find the average rate of change of C with respect to t over each time interval:
  - $[1.0, 2.0]$
  - $[1.5, 2.0]$
  - $[2.0, 2.5]$
  - $[2.0, 3.0]$ In each case, include the units.
- Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?

Exercise 20

Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30*ml* of ethanol (corresponding to two alcoholic drinks).

t (hours)	1	1.5	2	2.5	3
$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007

- Find the average rate of change of C with respect to t over each time interval:
  - $[1.0, 2.0]$
  - $[1.5, 2.0]$
  - $[2.0, 2.5]$
  - $[2.0, 3.0]$ In each case, include the units.
- Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?

Exercise 20

Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30*ml* of ethanol (corresponding to two alcoholic drinks).

t (hours)	1	1.5	2	2.5	3
$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007

- Find the average rate of change of C with respect to t over each time interval:
  - $[1.0, 2.0]$
  - $[1.5, 2.0]$
  - $[2.0, 2.5]$
  - $[2.0, 3.0]$ In each case, include the units.
- Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?