Graph Theory and Its Applications

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Lecture 4

Outline

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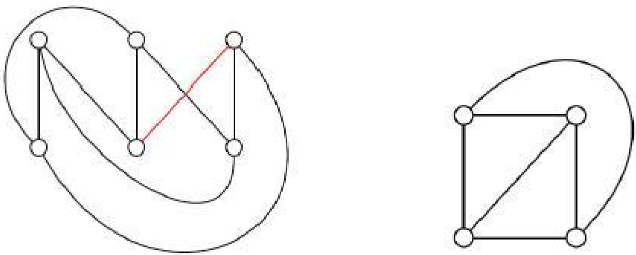
Planar graphs

When drawing connected graphs one is naturally lead to the question of crossing edges. One says that a graph is planar if it can be drawn (or represented) without crossing.



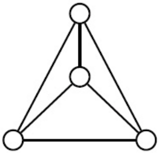
The above graphs represent $K_{3,3}$ (not planar) and K_4 (planar).

Definition

Let $G = (V, E)$ be a graph. A planar embedding of G is a picture of G in the plane such that the curves or line segments that represent edge intersect only at their end points. A graph that has plane embedding is called a planar graph.

Proposition

(Fary, 1948) Every planar graph can be represented in the plane using straight edges only.



Let $G = (V, E)$ be a graph embedded in the plane. Let,

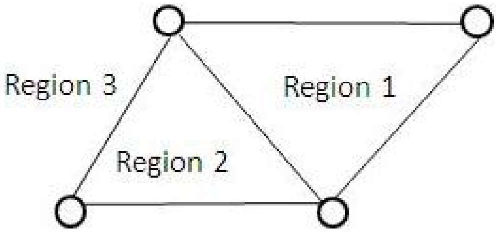
v = number of vertices in G.

e = number of edges in G.

c = number of components in G.

r = number of regions determined by the embedding of G.

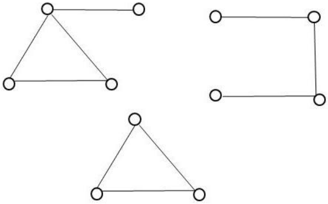
Example:



Theorem

Euler's Formula for every graph embedded in the plane is,
$v - e + r = c + 1$

Example:



$v = 11, e = 10, r = 3, c = 3 \Rightarrow 11 - 10 + 3 = 3 + 1.$

Proof

Follows...

Corollary

For every connected graph embedded in the plane

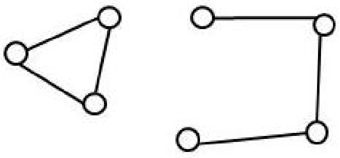
$$v - e + r = 2.$$

Theorem

Let G be a simple planar graph with at least two edges. Then

$$e \leq 3v - 6.$$

Example:

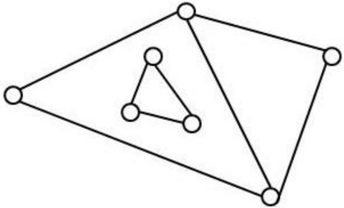


$e = 6, v = 7 \Rightarrow 6 \leq 3 \cdot 7 - 6 = 15.$

Lemma

(Edge-region inequality) Let G be a simple graph. Then, $2e \geq 3r$.

Example:



$e = 8, r = 4 \Rightarrow 16 = 2 \cdot 8 \geq 3 \cdot 4 = 12.$

Proof

Proof of the result $e \leq 3v - 6$.

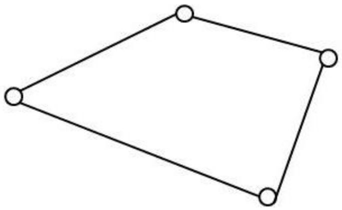
Follows...

Theorem

Let G be a planar graph with at least two edges, in which no cycle has length less than 4. Then,

$$e \leq 2v - 4.$$

Example:

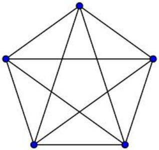


$e = 4, v = 4 \Rightarrow 4 \leq 2 \cdot 4 - 4 = 4.$

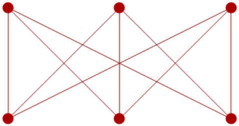
Theorem

The complete graph K_5 is not planar, and complete bipartite graph $K_{3,3}$ is not planar.

Examples:



By using $e \leq 3v - 6$ when, $e = 10, v = 5 \Rightarrow 10 \leq 15 - 6 = 9$, a contradiction.



By using $e \leq 2v - 4$ when, $e = 9, v = 6 \Rightarrow 9 \leq 2 \cdot 6 - 4 = 8$, a contradiction.

Corollary

The average degree of the vertices of a planar connected graph G is smaller than $6 - \dfrac{12}{n}$.

Corollary

In a planar (connected) graph there always exists a vertex v such that $d(v) \leq 5$.

Corollary

A planar graph can be colored with 6 colors (see later).

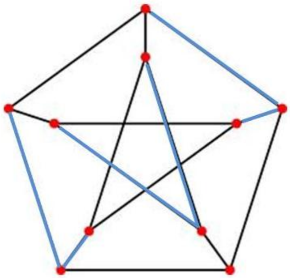
Definition

The graph $G^{'} = (V^{'}, E^{'})$ is said to be obtained from the graph $G = (V, E)$ by an edge bisection if $V^{'} = V \cup \{x\}$, $x \notin V$ and $E^{'} = (E - \{uv\}) \cup \{ux, vx\}$, $uv \in E$. A graph obtained from G by a sequence of zero or more edge bisectors is called a subdivision of G.

Theorem

A graph is a non planar if and only if it contains a subdivision of K_5 and $K_{3,3}$ as subgraph.

Example:



The Peterson graph is non planar, since it contains a subdivision of $K_{3,3}$. Why?