MAT 122 2.0 Calculus

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Lecture 4

Outline

1. Comparison Tests
   - The Comparison Test 1
   - Proof-Test 1
   - The Comparison Test 2
   - Proof-Test 2

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   - Proof-Test 1
   - The Comparison Test 2
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Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with positive terms.

1. If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$, for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.
2. If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$, for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

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- The first part says that if we have a series whose terms are smaller than those of a known convergent series, then our series is also convergent.

- The second part says that if we start with a series whose terms are larger than those of a known divergent series, then it too is divergent.

Examples

1. $\sum_{n=1}^{\infty} \frac{1}{n(n+7)}$ is convergent.

Solution

Let $a_n = \dfrac{1}{n(n+7)}$ and $b_n = \dfrac{1}{n^2}$, then $\dfrac{1}{n(n+7)} \leq \dfrac{1}{n^2}$, for all n.

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and $a_n \leq b_n$, for all n

$\Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+7)}$ is convergent.

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Examples

2. $\sum_{n=1}^{\infty} \frac{2n^2}{n^3+1}$ is divergent.

Solution

Let $a_n = \dfrac{2n^2}{n^3+1}$ and $b_n = \dfrac{1}{n}$, then $\dfrac{2n^2}{n^3+1} \geq \dfrac{1}{n}$, for all n.

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $a_n \geq b_n$, for all n

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Proof of the Test 1-Part (1)

Suppose $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$, for all n.

Let $S_n = \sum_{r=1}^{n} a_r$ and $T_n = \sum_{r=1}^{n} b_r$.

Then, $\lim_{n\to\infty} T_n = t$, for some $t \in \mathbb{R}$.

Since both series are have positive terms, $\{S_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ are increasing sequences.

Since $\lim_{n\to\infty} T_n = t$, $T_n \leq t$, for all n.

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Proof Contd...

Since $a_r \leq b_r$, $S_n \leq t$, for all n.

i.e. $\{S_n\}_{n=1}^{\infty}$ is increasing and bounded above.

$\lim_{n\to\infty} S_n$ exists.

Hence $\sum_{n=1}^{\infty} a_n$ is convergent.

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Proof of the Test 1-Part (2)

Now suppose $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$, for all n,

then $T_n = \sum_{r=1}^{n} b_r \longrightarrow \infty$ (Since $\{T_n\}_{n=1}^{\infty}$ is increasing).

But $a_r \geq b_r$ so $S_n \geq T_n > 0$.

Thus, $S_n \longrightarrow \infty$.

$\therefore \sum_{n=1}^{\infty} a_n$ is diverges.

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Test

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If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where $L > 0$ is a finite number,

then either both series converge or both series diverge.

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Examples

1. Test the series $\sum_{n-1}^{\infty} \frac{1}{3^n-1}$ for convergence or divergence.

Solution

Let $a_n = \frac{1}{3^n-1}$ and $b_n = \frac{1}{3^n}$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3^n}{3^n-1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{3^n}} = 1 > 0$

Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent geometric series, the given series converges by the The Limit Comparison Test.

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Examples

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{3n^2+2n}{n^3+1}$ converges or diverges.

Solution

Let $a_n = \frac{3n^2+2n}{n^3+1} = \frac{n^2(3+\frac{2}{n})}{n^3(1+\frac{1}{n^3})}$. Then $\frac{a_n}{\frac{1}{n}} = \frac{(3+\frac{2}{n})}{(1+\frac{1}{n^3})}$. Thus, $b_n = \frac{1}{n}$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3+\frac{2}{n})}{(1+\frac{1}{n^3})} = 3 > 0.$

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Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the given series is divergent by the The Limit Comparison Test.

Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where $L > 0$ is a finite number.

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, $m \leq \frac{a_n}{b_n} \leq M$, for all $n > n_0$, for some positive numbers m and M

$\therefore mb_n \leq a_n \leq Mb_n$, for all $n > n_0$.

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Proof Contd..

If $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} Mb_n$. Thus $\sum_{n=1}^{\infty} a_n$ converges by the Comparison Test 1-Part (1).

If $\sum_{n=1}^{\infty} b_n$ diverges, so does $\sum_{n=1}^{\infty} mb_n$ and by the Comparison Test 1-Part (2). shows that $\sum_{n=1}^{\infty} a_n$ diverges.

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