Graph Theory and Its Applications

Dr. G.H.J. Lanel

Lecture 5

Outline

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1. Trees
   - Introduction
   - Summary of basic properties
   - Rooted trees
   - Some terminologies
   - Definitions
   - Counting trees
   - Spanning trees

Definition

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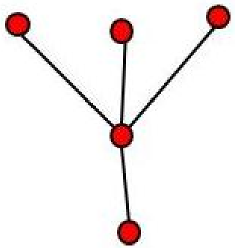

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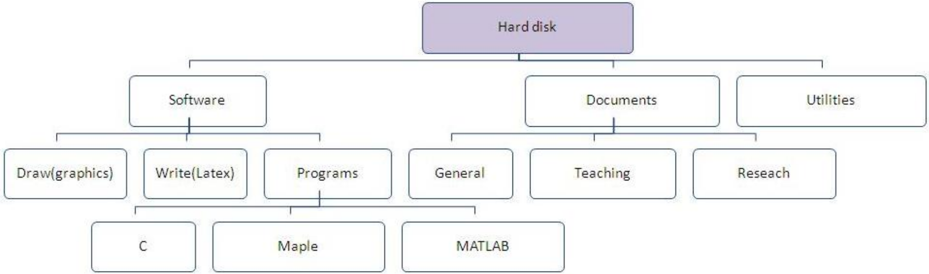

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In an undirected tree, a leaf is a vertex of degree 1.

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An acyclic graph is called a forest.

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Proposition

A forest G on n vertices has $n - c(G)$ edges, where $c(G)$ is the number of components of G.(True for any graph)

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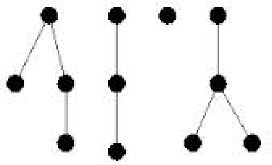

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For a graph $G = (V, E)$ of order $n = |V|$, the following are equivalent

1. *G is acyclic and has $n - 1$ edges.*
2. *G is connected and has $n - 1$ edges.*
3. *G is connected and acyclic.*
4. *$\forall u, v \in V$ there is one and only one path from u to v.*
5. *G is acyclic and adding an edge creates one and only one cycle.*
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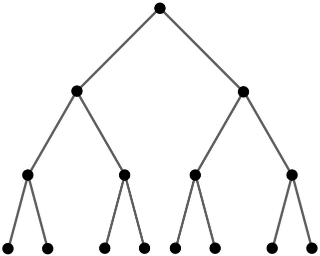

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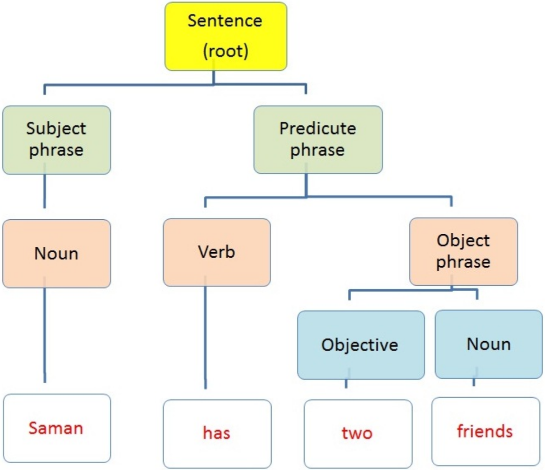

1. In a rooted tree, the depth or level of a vertex v is its distance from the root.
2. The height of a rooted tree is the length of a longest path from the root.
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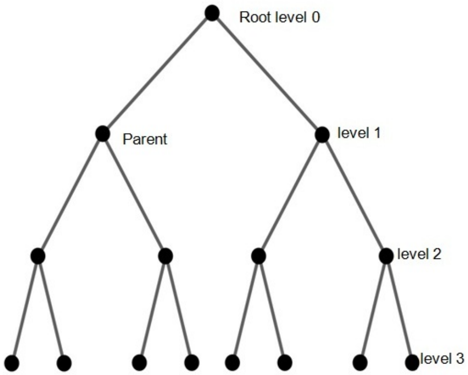

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The eccentricity $\varepsilon(u) = max_{v \in V} d(u, v)$ of a node is the maximum distance to any node $v \in V$.

Proposition

Let T be a tree and let T' be the tree obtained by removing all its leafs, then $\varepsilon(T') = \varepsilon(T) - 1$ for all nodes of T'.

Proposition

The center of a tree is a single node or a pair of adjacent nodes.

Proof

By induction and using the previous proposition. Show that the center does not change.

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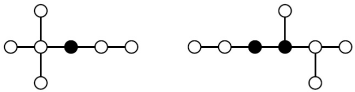

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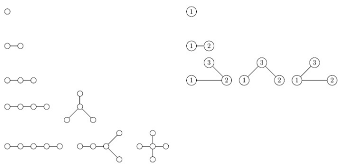

n	Count
1	$\to 1$
2	$\to 1$
3	$\to 3$
4	$\to 16$
5	$\to 125$

The following theorem of Cayley gives the exact formula.

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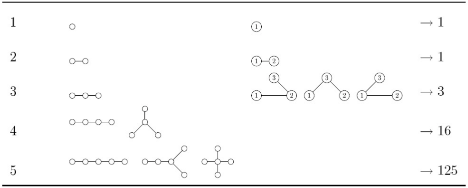

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