AMT 223 1.0 Discrete Mathematics (General Degree)

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Semester 2 - 2018

Outline

The Multiplication Principle

Recall: For a set A, $|A|$ is the cardinality of A (Number of elements of A). For a pair of sets A and B, $A \times B$ denotes their Cartesian product: $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

The Multiplication Principle - Set Formulation

If A and B are finite sets, then: $|A \times B| = |A| \cdot |B|$.

The Multiplication Principle - Task Formulation

If $A_1, A_2, ..., A_m$ are finite sets, then
$|A_1 \times A_2 \times ... \times A_m| = |A_1| \; |A_2| ... |A_m|$

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Examples

Set Formulation

Let $A = \{2, 4\}$ and $B = \{1, 2, 5\}$. Then $|A| = 2$ and $| B |= 3$. Furthermore, $A \times B$ consists of the pairs (2, 1), (2, 2),(2, 5), (4, 1), (4, 2), (4, 5) and has cardinality $2 \cdot 3 = 6$

Task Formulation

- Consider the task performed by a man getting dressed in the morning and having to decide what to wear.
- Suppose that he owns a green shirt, a red shirt and a yellow shirt; brown trousers and charcoal trousers; and a green tie, a red tie and a yellow tie.
- He wants to wear a shirt, trousers, and a tie, and his sense of aesthetics dictates that the tie have a color different from that of the shirt.
- In how many ways can he get dressed; that is, how many different outfits can he construct subject to these constraints?

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The Addition Principle

Addition Principle - Set Formulation

If A and B are finite sets that are disjoint (meaning $A \cap B = \emptyset$), then $\mid A \cup B \mid = \mid A \mid + \mid B \mid$

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If $A_1, ..., A_m$ are finite sets that are pairwise disjoint, meaning $A_i \cap A_j = \emptyset$, for all $i, j \in \{1, ..., m\}$, then $\mid A_1 \cup A_2 \cup ... \cup A_m \mid = \mid A_1 \mid + \mid A_2 \mid + ... + \mid A_m \mid$

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Set Formulation

- Let A be the set of prime numbers between 10 and 30, and let B be the set of multiples of 7 in the same interval.
- It is easy to check that $|A| = 6$ and $|B| = 3$.
- Since no multiple of 7 greater than 7 is prime, A and B are disjoint sets.
- Thus the cardinality of $A \cup B$ (the set consisting of numbers between 10 and 30 that are either prime or multiples of 7) is 6 + 3 = 9.

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- A woman needs to decide what to wear.
- She can wear a skirted suit with a white blouse, and she owns three skirted suits.
- Alternatively, she can wear one of her six dresses, or she can choose any of four slacks along with her only sweater.
- How many different ways can she get dressed?

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The Over-counting Principle

- Many times when trying to apply the multiplication principle or addition principle to a counting problem, we count more than we should.
- We might include in our count some elements not in the set we are counting, or we might end up counting some elements more than once.
- Obviously, then, the answer we obtain will be wrong.
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- If our answer is too large by an amount k because we counted k items that we should not have counted, then we can simply subtract k to get the correct answer.
- If our answer is too large because we counted k of the elements twice when we wanted to count them only once, then again we can simply subtract k to obtain the correct answer.
- If our answer is too large by a factor m because we counted each element we wanted to count, not once, but m times, then we can simply divide by m to obtain the correct answer.
- Actually, for many problems, we *purposely* over-count, and then correct the over-count, as the neatest way to obtain the answer.
- We formalize this discussion as a theorem.

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Theorem (The Over-counting Principle)

Lat A be a finite at whose cardinality is to be computed.

1. *If $A \subseteq B$ and $| B |= n$ and $| B - A |= k$, then $| A |= n - k$.*
2. *If n is the answer obtained when counting the number of elements in a set A, except that in this count exactly k of the elements were counted twice, then $| A |= n - k$*
3. *If n is the answer obtained when counting the number of elements in a set A, except that in this count each element was counted exactly m times, then $| A |= n/m$.*

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By applying part (2), we obtain the following rule for finding the cardinality of the union of two sets that are not necessarily disjoint.

It is a generalization of the multiplication principle - task formulation.

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If $A = B \cup C$, then $\mid A \mid = \mid B \mid + \mid C \mid - \mid B \cap C \mid$

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If $A = B \cup C$*, then* $\mid A \mid = \mid B \mid + \mid C \mid - \mid B \cap C \mid$

Proof

To count the number of elements of A, we count the elements of B (they are all in A), and we count the elements of C (they are all in A), but then we note that we have counted the elements that were in both B and C (in other words, in $B \cap C$) twice.

Since we only wanted to count each element once, we must subtract the over-count, that is, the number of elements in $B \cap C$.

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How many bit strings of length $n \geq 2$ either begin or end with a 1?

Solution.

- A bit string that begins with a 1 can be constructed by choosing the bits one at a time, working from left to right.
- Since there are two choices for each bit other than the first, the multiplication principle tells us that there are $1 \cdot 2 \cdot 2 \cdots 2 = 2^{n-1}$ such strings.
- Similarly, there are $2 \cdot 2 \cdot 2 \cdot 1 = 2^{n-1}$ bit strings of length n that end with a 1.

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Solution Ctd.

These two sets of strings are not disjoint, however, since a bit string can both begin and end with a 1, so that if we were simply to add 2^{n-1} and 2^{n-1}, we would have over-counted by the number of bit strings that both begin and end with a 1.

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Thus by theorems, we obtain the answer $2^{n-1} + 2^{n-1} - 2^{n-2} = 3 \cdot 2^{n-2}$.

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Example 2

- A pair of distinct positive integers is called an ”amicable pair” if the sum of all the proper divisors of the first is equal to the second, and the sum of all the proper divisors of the second is equal to the first.
- For example, 220 and 284 form an amicable pair, since the sum of all proper divisors of 220 is $1+2+4+5+10+11+20+22+44+55+110=284$ and the sum of all the proper divisors of 284 is $1+2+4+71+142=220$.

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- A naive algorithm for finding all such pairs within a given range would proceed as follows:

procedure *amicable*(n : positive integer)
{this algorithm finds and prints all amicable pairs (i, j) with $1 \leq i < j \leq n$
for $i \leftarrow 1$ **to** $n - 1$ **do**
for $j \leftarrow i + 1$ **to** n **do**
if (sum of proper divisors of i) = j
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- Suppose that we want to estimate the number of steps required by this algorithm to print all the amicable pairs (i, j) with $1 \leq i < j \leq n\}$, so that we can tell whether running this algorithm is feasible for a particular n.

- We will compute the number of times the condition in the **if** clause is checked.

- The number of times the condition is checked is equal to the number of pairs (i, j) with $1 \leq i < j \leq n\}$.

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- Then by the multiplication principle, since there are n choices for the value of i, followed by $n-1$ choices for the value of j once i has been chosen, there are $n(n-1)$ such pairs.
- By symmetry, in exactly half of these pairs is $i < j$ (in the remaining half, $j < i$).
- Thus we have over-counted what we really wanted by a factor of 2.
- Therefore, the number of pairs (i, j) with $1 \le i < j \le n$, that is, the number of times the condition is checked, is equal to $n(n-1)/2$.

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- Thus we have over-counted what we really wanted by a factor of 2.
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