MAT 122 2.0 Calculus

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Lecture 5

Outline

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1. Alternating Series
   - Alternating Series
   - The Alternating Series Test
   - Absolute Convergence
   - Conditionally Convergence

2. Ratio and Root Tests
   - Ratio Tests
   - Root Tests

Consider the series whose terms are not necessarily positive.

- An **Alternating Series** is a series whose terms are alternately positive and negative.

Examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

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If the alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots, b_n > 0$$

with,

1. $b_{n+1} \leq b_n$, for all n
2. $\lim_{n\to\infty} b_n = 0$, (the limit of the n^{th} term of the series is 0).

then the series is convergent.

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The alternating harmonic series,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

which satisfies,

1. $\frac{1}{n+1} < \frac{1}{n} \Rightarrow b_{n+1} < b_n$ and
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Example 2:

The series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3n}{4n-1}$ is alternating.

But $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4-\frac{1}{n}} = \frac{3}{4}$

So the condition (2) is not satisfied, the limit of the nth term is not 0.

So the series diverges.

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Example 3:

Test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ for convergence or divergence.

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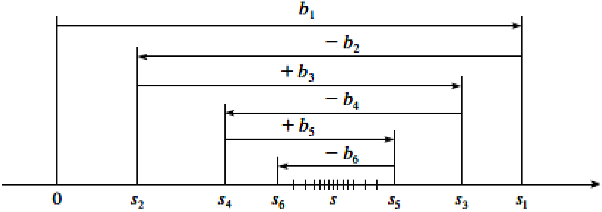
We construct the following



$s_2 = b_1 - b_2 \geq 0$, since $b_2 \leq b_1$

$s_4 = s_2 + (b_3 - b_4) \geq s_2$, since $b_4 \leq b_3$

$s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2}$, since $b_{2n} \leq b_{2n-1}$, for all n

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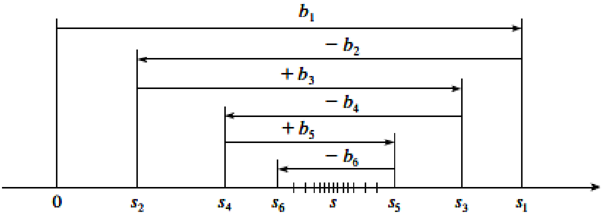

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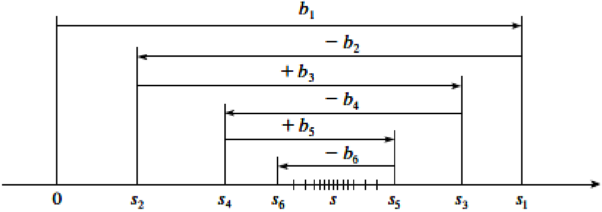

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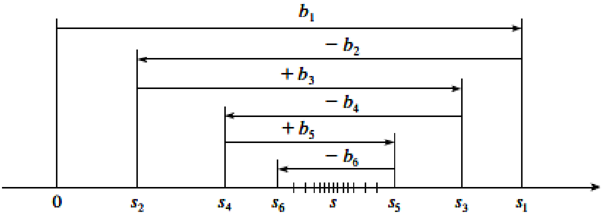

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Proof Contd.

Thus $0 \leq S_2 \leq S_4 \leq S_6 \leq \cdots \leq S_{2n} \leq \cdots$

and we can also write,

$S_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$

Every term in brackets is positive, so $S_{2n} \leq b_1$, for all n.

Therefore, the sequence $\{S_{2n}\}$ of even partial sums is increasing and bounded above.

$\therefore \lim_{n\to\infty} S_{2n} = S$, so the series is convergent.

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Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

Definition

A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Notice that if $\sum a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

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Theorem

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Proof :

Consider the the inequality,

$$0 \leq a_n + |a_n| \leq 2\,|a_n|$$

if $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, so $\sum 2\,|a_n|$ is convergent.

Therefore, by the Comparison Test, $\sum (a_n + |a_n|)$ is convergent.

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$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

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The series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because,

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But it is not absolutely convergent, because the corresponding series of absolute value is,

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which is the harmonic series and is therefore divergent.

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$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series and is therefore divergent.

Definition

A series $\sum a_n$ is said to converge conditionally if $\sum a_n$ is converges while $\sum |a_n|$ diverges (*Not coverage absolutely*).

Eg. 01.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots \quad \text{is converge conditionally.}$$

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Outline

1. Alternating Series
   - Alternating Series
   - The Alternating Series Test
   - Absolute Convergence
   - Conditionally Convergence

2. Ratio and Root Tests
   - Ratio Tests
   - Root Tests

The Ratio Test is effective with factorials and with combinations of powers and factorials.

The Ratio Test

1. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore it is convergent.
2. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

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Eg. 01: Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ for absolute convergence.

Solution: Using The Ratio Test with $a_n = (-1)^n \frac{n^3}{3^n}$

$$
\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right| \\
&= \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \\
&= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 \\
&= \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1 \text{ as } n \to \infty
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Thus, by the Ratio Test, the given series is absolutely convergent and therefore convergent.

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Eg. 02: Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution: Since the terms $a_n = \frac{n^n}{n!}$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n$$

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Since $e > 1$, the given series is divergent by the Ratio Test.

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Proof of the Test:

Part (1): The idea is to compare the given series with a convergent geometric series. Since $L < 1$, we choose a number r such that $L < r < 1$. Since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

the ratio $\left| \frac{a_{n+1}}{a_n} \right|$ will eventually be less than r that is, there exists an integer N such that

$$\left| \frac{a_{n+1}}{a_n} \right| < r, \text{ whenever } n \geq N$$

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Proof Contd.

Putting n successively equal to $N, N+1, N+2, \leftarrow$ in (1), we can obtain

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Now the series

$$\sum_{k=1}^{\infty} |a_N| r^k = |a_N| r + |a_N| r^2 + |a_N| r^3 + \cdots$$

is convergent because it is a geometric series with $0 < r < 1$. So the inequality (2) together with the Comparison Test, show that the series

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(*Recall that a finite number of terms doesn't affect convergence.*)
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Part (2): If

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will eventually be greater than 1; that is, there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1, \text{ whenever } n \geq N$$

This means that $|a_{n+1}| > |a_n|$, whenever $n \geq N$ and so

$$\lim_{n\to\infty} a_n \neq 0$$

Therefore, $\sum a_n$ diverges by the Test for Divergence.

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whereas for the divergent series $\sum \frac{1}{n}$ we have

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Therefore, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the series $\sum a_n$ might converge or it might diverge. In this case the Ratio Test fails and we must use some other test.

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The Root Test is used only if powers are involved.

The Root Test

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

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Example.

Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

Solution:

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1 \text{ as } n \to \infty$$

Thus, the given series converges by the Root test.

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