Complex Variables

Dr. G.H.J. Lanel

Lecture 6

Outline

Outline

1. Applications
   - Vector Fields
   - Potential Functions
   - Steady-State Fluid Flow

Vector Fields

A vector field $F(x, y) = P(x, y)i + Q(x, y)j$ in a domain D can also be expressed in the complex form

$$F(x, y) = P(x, y) + iQ(x, y)$$

Recall that div $F = \partial P/\partial x + \partial Q/\partial y$ and $curlF = (\partial Q/\partial x - \partial P/\partial y)k$. If we require both of them are zeros, then

$$\frac{\partial P}{\partial x} = -\frac{\partial Q}{\partial y} \quad \text{and} \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

THEOREM 7

(i) Suppose that $F(x, y) = P(x, y) + Q(x, y)$ is a vector field in a domain D and $P(x, y)$ and $Q(x, y)$ are continuous and have continuous first partial derivatives in D. If div $\mathbf{F} = 0$ and curl $\mathbf{F} = \mathbf{0}$, then complex function

$$g(z) = P(x, y) - iQ(x, y)$$

is analytic in D.

(ii) Conversely, if $g(z)$ is analytic in D, then $\mathbf{F}(x, y) = \overline{g(z)}$ defined a vector field in D for which div $\mathbf{F} = 0$ and curl $\mathbf{F} = \mathbf{0}$.

Theorem 7: Proof

Proof

If $u(x, y)$ and $v(x, y)$ denote the real and imaginary parts of $g(z)$, then $u = P$ and $v = -Q$. Then

$$\frac{\partial u}{\partial x} = -\frac{\partial(-v)}{\partial y} \ , \ \frac{\partial u}{\partial y} = \frac{\partial(-v)}{\partial x}; \text{ that is,}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Equations in (2) are the Cauchy-Riemann equations for analyticity.

Example 1

The vector field $F(x, y) = (-kq/|z - z_0|^2)(z - z_0)$ may be interpreted as the electric field by a wire that is perpendicular to the z-plane at $z = z_0$ and carries a charge of q coulombs per unit length.

The corresponding complex function is

$$g(z) = \frac{-kq}{|z-z_0|^2}\overline{(z - z_0)} = \frac{-kq}{z-z_0}$$

Since g(z) is analytic for $z \neq z_0$, $divF = 0$, $curlF = 0$

Example 2

The complex function $g(z) = Az$, $A > 0$, is analytic in the first quadrant and therefore gives rise to the vector field

$$V(x, y) = \overline{g(z)} = Ax - iAy$$

which satisfies div V=0,curl V=0

Potential Functions

Suppose that $F(x, y)$ is a vector field in a simply connected domain D with div $\mathbf{F} = 0$ and curl $\mathbf{F} = 0$.

By Theorem, the analytic function $g(z) = P(x, y) - iQ(x, y)$ has an antiderivative

$G(z) = \phi(x, y) + i\psi(x, y) \longrightarrow (4)$

in D, which is called a complex potential for the vector filed **F**.

Note that $g(z) = G'(z) = \frac{\partial\phi}{\partial x}(x, y) + i\frac{\partial\psi}{\partial x}(x, y)$

$= \frac{\partial\phi}{\partial x}(x, y) - i\frac{\partial\phi}{\partial y}(x, y)$

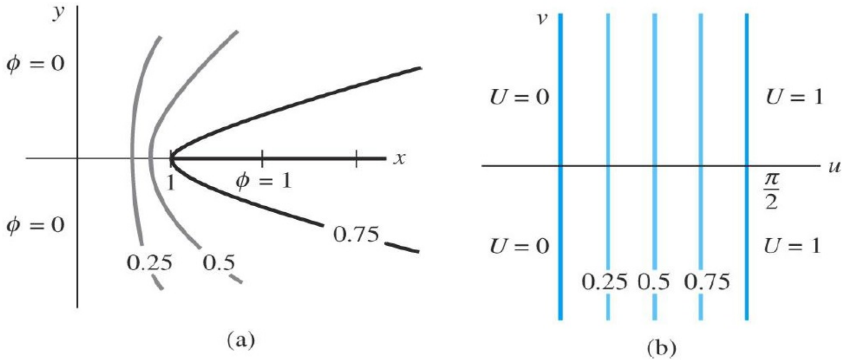
and so $\frac{\partial\phi}{\partial x} = P, \frac{\partial\phi}{\partial y} = Q$ (5)

Therefore $\mathbf{F} = \Delta\phi$, and the harmonic function ϕ is called a (real) potential function of **F**.

Example 3

The potential ϕ in the half-plane $x \geq 0$ satisfies the boundary conditions $\phi(0, y) = 0$ and $\phi(x, 0) = 1$ for $x \geq 1$.

See the following Figure (a). Determine a complex potential, the equipotential lines, and the field **F**.



Example 3: Cont...

Solution

We knew the analytic function $z = sin(w)$ maps the strip $0 \leq u \leq \pi/2$ in the w-plane to the region R in question. Therefore $f(z) = sin^{-1}z$ maps R onto the strip, and the above Figure (b) shows the transferred boundary conditions.

The simplified Dirichlet problem has the solution $U(u, v) = (2/\pi)u$, and so $\phi(x, y) = U(sin^{-1}z) = Re((2/\pi)sin^{-1}z)$ is the potential function on D, and $G(z) = (2/\pi)sin^{-1}z$ is a complex potential for **F**.

Example 3: Cont...

Note that the equipotential lines $\phi = c$ are the images of the equipotential lines $U = c$ in the w-plane under the inverse mapping $z = sin(w)$. We found that the vertical lines $u = a$ is mapped onto a branch of the hyperbola

$$\frac{x^2}{\sin^2 a} - \frac{y^2}{\cos^2 a} = 1$$

Example 3: Cont...

Since the equipotential lines $U = c$, $0 < c < 1$ is the vertical line $u = \pi/2c$, it follows that the equipotential lines $\phi = c$ is the right branch of the hyperbola

$$\frac{x^2}{\sin^2(\pi c/2)} - \frac{y^2}{\cos^2(\pi c/2)} = 1$$

Since $\mathbf{F} = \overline{G'(z)}$ and $(d/dz)\sin^{-1} z = 1/(1 - z^2)^{1/2}$,

then $\mathbf{F} = \frac{2}{\pi}\overline{\frac{1}{\pi(1-z^2)^{1/2}}} = \frac{2}{\pi}\frac{1}{\pi(1-\bar{z}^2)^{1/2}}$

Steady-State Fluid Flow

The vector $V(x,y) = P(x,y) + iQ(x,y)$ may also be expressed as the velocity vector of a two-dimensional steady-state fluid flow at a point (x,y) in a domain D. If div $\mathbf{V} = 0$ and curl $\mathbf{V} = 0$, $\mathbf{V}$ has a complex velocity potential

$$G(z) = \phi(x,y) + \psi(x,y)$$

that satisfies

$$\overline{G'(z) = \mathbf{V}}$$

Here special importance is placed on the level curves $\psi(x, y) = c$. If $z(t) = x(t) + iy(t)$ is the path of a particle, then

$$\frac{dx}{dt} = P(x, y), \frac{dy}{dt} = Q(x, y) \quad (6)$$

Hence

$$dy/dx = Q(x, y)/P(x, y) \text{ or}$$

$$-Q(x, y)dx + P(x, y)dy = 0.$$

Since *div* $\mathbf{V} = 0$ *implies* $\frac{\partial(-Q)}{\partial y} = \frac{\partial P}{\partial x}$

and by the Cauchy-Riemann equations

$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = -Q$ and $\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = P$

and all solutions of (6) satisfy $\psi(x, y) = c$.

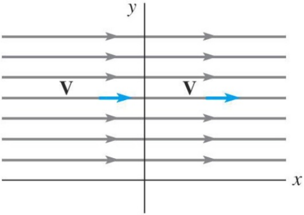
The function $\psi(x, y)$ is called a stream function and the level curves $\psi(x, y) = c$ are streamlines for the flow.

Note that: Stream line is a line, which is everywhere tangent to the velocity vector at a given instant. Stream Function is defined as the scalar function of space and time such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction

Example 4

The uniform flow in the upper half-plane is defined by $V(x, y) = A(1, 0)$, where A is a fixed positive constant. Note that $|V| = A$, and so a particle in the fluid moves at a constant speed.

A complex potential for the vector field is $G(z) = Az = Ax + iAy$, and so the streamlines are the horizontal lines $Ay = c$. See the following Figure(a). Note that the boundary $y = 0$ of the region is itself streamline.



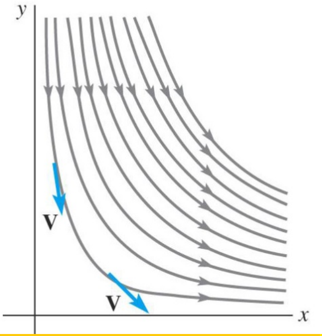
(a)

Example 5

The analytic function $G(z) = z^2$ gives rise to the vector field

$$\mathbf{V}(x, y) = \overline{G'(z)} = (2x, -2y)$$

in the first quadrant. Since $z^2 = x^2 - y^2 + i(2xy)$, the stream function is $\psi(x, y) = 2xy$ and the streamlines are the hyperbolas $2xy = c$. See the following Figure(b).



Theorem 8

THEOREM 8

Suppose that $G(x) = \phi(x, y) + i\psi(x, y)$ is analytic in a region R and $\psi(x, y)$ is continuous on the boundary of R. Then $\mathbf{V}(x, y) = \overline{G'(z)}$ defined an irrotational and incompressible fluid flow in R. Moreover, if a particle is placed is placed inside R, its path $z = z(t)$ remains in R.

Example 6

The analytic function $G(z) = z + 1/z$ maps the region R in the upper half-plane and outside the circle $|z| = 1$ onto the upper half-plane $v \geq 0$. The boundary of R is mapped onto the u-axis, and so $v = \psi(x, y) = y - y/(x^2 + y^2)$ is zero on the boundary of R.

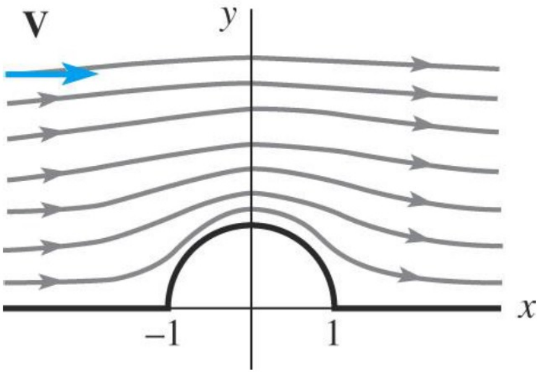
The following Figure shows the streamlines. The velocity field is given by

$$\overline{G'(z)} = 1 - 1/\overline{z^2}, \text{ and so}$$

$$\overline{G'(re^{i\theta})} = 1 - \frac{1}{r^2}e^{2i\theta}$$

Example 6: Cont...

It follows that $V \cong (1, 0)$ for large values of r, and so the flow is approximately uniform at large distance from the circle $|z| = 1$. The resulting flow in R is called flow around a cylinder.



Example 7

The analytic function $f(w) = w + Ln(w) + 1$ maps the upper half-plane $v \geq 0$ to the upper half-plane $y \geq 0$, with the horizontal line $y = \pi$, $x \leq 0$, deleted.

See Example 4 in Section Schwarz-Christoffel Transformations. If $G(z) = f^{-1}(z) = \phi(x, y) + i\psi(x, y)$, then $G(z)$ maps R onto the upper half-plane and maps the boundary of R onto the u-axis. Therefore $\psi(x, y) = 0$ on the boundary of R.

Example 7: Cont...

It is not easy to find an explicit formula for $\psi(x, y)$. The streamlines are the images of the horizontal lines $v = c$ under $z = f(w)$. If we write $w = t + ic$, $c > 0$, then the streamlines can be

$$z = f(t + ic) = t + ic + Ln(t + ic) + 1, \text{ that is,}$$

$$x = t + 1 + \frac{1}{2} log_e(t^2 + c^2), y = c + Arg(t + ic)$$

See the following figure.