# Complex Variables

Dr. G.H.J. Lanel

Lecture 1

# Outline

### Theorem

*Let $\gamma$ be a simple closed curve traversed once anti-clockwise, and let $D$ be the domain enclosed by $\gamma$. Let be a function which is analytic on $\gamma \cup D$ except at a finite number of singularities $z_1, ..., z_n$ in $D$.*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z).$$

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# Example

- Take $f(z) = \frac{1}{z}$, this is has a simple pole at 0 with residue 1 (why?), and is analytic everywhere else in the complex plane $\mathbb{C}$.
- So if $\gamma$ is a simple closed curve traversed once anti-clockwise that enclose the origin,
- e.g: $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$ then $\int_\gamma f(z)dz = 2\pi i$ (one can of course also obtain this from the Cauchy integral formula, which is a special case of the residue theorem).

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# Example cont...

- If however $\gamma$ is a simple closed curve that does not enclose the origin,
- e.g: $\gamma(t) = 3 + e^{it}$ for $0 \leq t \leq 2\pi$ then $f(z)$ is analytic on and inside $\gamma$ , and then by Cauchy's theorem $\int_\gamma f(z)dz = 0$.
- If instead $\gamma$ is a simple closed curve traversed clockwise that encloses the origin,
- e.g: $\gamma(t) = e^{-it}$ for $0 \leq t \leq 2\pi$ then we have instead $\int_\gamma f(z)dz = -1$, since the integral along $\gamma$ is the negative of the integral along $-\gamma$ (the reversal of $\gamma$).

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# Example cont...

- Finally, if $\gamma$ is a non-simple closed curve which encloses the origin twice,
- e.g: $\gamma(t) = e^{it}$ for $0 \leq t \leq 4\pi$ then we have $\int_\gamma f(z)dz = 4\pi i$ , because we can break $\gamma$ into sum of two smaller simple closed curves,each of which enclose the origin.
- Note that in each of these cases $\gamma$ does not actually pass through the singularity, but only goes around it.
- If $\gamma$ passed through the singularity then the integral would only make sense as a principal value integral, as the integrand would be going to infinity at a point on $\gamma$.

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- These examples show that in order to compute an integral $\int_\gamma f(z)dz$ when $\gamma$ is closed and $f$ has a finite number of singularities, it is not always enough to simply just add up all the residues of $f$ inside $\gamma$ and multiply by $2\pi i$;
- the winding number of $\gamma$ around the singularities makes a difference.
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### Theorem

*Let D be a simply connected domain, and f be a function analytic on D except at a finite number of singularities $z_1, z_2, ..., z_n$. Let $\gamma$ be a closed (but not necessarily simple) curve in D which does not pass through any of the singularities $z_1, z_2, ..., z_n$,*

*Then we have*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} W(\gamma, z_k) Res_{z=z_k} f(z);$$

in other words, we add up all the residues as before, but we multiply each residue by the winding number of $\gamma$ around $z_k$ (which could be a positive integer, negative integer, or zero).

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- The winding number of a closed curve $\gamma$ around a point $z_0$ which does not lie on $\gamma$ is defined by

$$W(\gamma, z_0) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

- The winding is not defined if $z_0$ passes through $\gamma$.
- The winding number is always an integer, and (informally speaking) counts how many times $\gamma$ winds anti-clockwise around $z_0$.
- It is additive: if $\gamma_1, \gamma_2$ are two closed curves with the same starting point, then $W(\gamma_1 + \gamma_2, z_0) = W(\gamma_1, z_0) + W(\gamma_2, z_0)$. Similarly we have $W(-\gamma, z_0) = -W(\gamma, z)$.

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### Definition (Holomorphic)

Holomorphic function is a complex-valued function of one or more complex variables that is complex differentiable in a neighborhood of every point in its domain.

### Definition (Meromorphic)

Relating to or being a function of a complex variable that is analytic everywhere in a region except for singularities at each of which infinity is the limit and each of which is contained in a neighborhood where the function is analytic except for the singular point itself.

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## Definition

Let $D$ be a domain. We say that a function $f : D \to C$ is meromorphic on $D$ if it only has a finite number $z_1, ..., z_n$ of singularities on $D$, and all of those singularities are poles or removable singularities (i.e. no essential singularities; all Laurent series around a singularity have only a finite number of singular term).

A meromorphic function with no singularities at all is called `analytic` or `holomorphic`.

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# Example

- In the disk $|z| < 1$, the function $\frac{1}{z(z-2)(z-\frac{1}{2})}$ is meromorphic, with poles at 0 and $\frac{1}{2}$ (the pole at 2 is irrelevant since it is outside the domain of interest).
- But the function $e^{\frac{1}{z}}$ is not meromorphic because it has an essential singularity at 0.
- However, the function $e^{\frac{1}{(z-2)}}$ is meromorphic on the disk , indeed it is holomorphic here.

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Note that:

- Meromorphic functions of course have a residue at every pole, but it is sometimes difficult to compute.
- But if instead of considering a meromorphic function $f$, one considers another function called the *logarithmic derivative* of $f$, the residues become very easy (and useful) to compute.

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## Definition

Let $f$ be a meromorphic function on a domain $D$ with only finitely many zeroes. we define the *logarithmic derivative* of $f$ to be the function $\frac{f'(z)}{f(z)}$.

- The logarithmic derivative is not defined when $f$ has a singularity(clearly), but also not defined when $f$ has a zero (since then we are dividing by zero).
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- Everywhere else, though, the logarithmic derivative is defined and is even analytic (because we are dividing one analytic function by another non-zero analytic function).

- The reason why $\frac{f'(z)}{f(z)}$ is called logarithmic derivative is because, formally it is what the chain rule would say the derivative of $log f(z)$ (unfortunately $log f(z)$ is multi-valued and so this isn't quite correct but it is fairly close to accurate and so calling this the logarithmic derivative is not a bad idea.)
- example: The logarithmic derivative of $e^{z^2}$ is $2z$. the logarithmic derivative of $\frac{z}{z-1}$ is $\frac{1}{z} - \frac{1}{z-1}$.

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# End!