Complex Variables

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Lecture 8

Outline

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Theorem

Let f be a meromorphic function on a domain D with only finitely many zeroes.

If f has a pole at z_0 with order m, then the logarithmic derivative $\frac{f'(z)}{f(z)}$ has a simple pole at z_0 with residue $-m$.

If instead f has a zero at z_0 with order m, then the logarithmic derivative $\frac{f'(z)}{f(z)}$ has a simple pole at z_0 with residue m.

Proof

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- The case when f has a zero of order m at z_0 is very similar and is left to the reader as the Assignment 2.
- In fact, poles and zeroes are opposites of each other in many ways; one can think of a pole of order m as a zero of order $-m$ or conversely a zero of order m as a zero of order $-m$.

Example

- The function $f = (z-2)^3 e^z/(z-1)^4$ has a triple zero at 2, and hence its logarithmic derivative f'/f has a simple pole at 2 with residue 3.
- Similarly it has a simple pole at 1 with residue -4.

Combining the new results with the residue theorem, we obtain the following corollary;

Corollary

Let D be a simply connected domain, and let f be a meromorphic function on D with finitely many zeroes and no removable singularities (i.e the only singularities are poles). Then for any simple closed curve γ *traversed once anticlockwise which does not pass through any zero or pole of f, the quantity*

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz$$

is equal to the number of zeroes inside γ*, minus the number of poles inside* γ

Example

- Consider the function $f = (z-2)^3 e^z/(z-1)^4$ mentioned earlier; this has a triple zero at 2, a quadruple pole at 1, and no other zeroes or poles.
- Thus if γ is the circle $\gamma(t) = 3e^{it} : 0 \leq t \leq 2\pi$, then $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = 3 - 4 = -1.$
- If γ' is the circle $\gamma'(t) = \frac{3}{2}e^{it} : 0 \leq t \leq 2\pi$, then $\frac{1}{2\pi i}\int_{\gamma'} \frac{f'(z)}{f(z)}dz = -4.$

- This is already a useful formula. It means that one can count the number of zeroes and poles of a meromorphic function f inside a region by integrating its algorithmic derivative on the boundary of that region.
- More precisely, the formula does not count the number of zeroes or the number of poles separately, but rather counts difference between the two.
- This is a fact of life; if a function has both zeroes and poles or just look at the zeroes.

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- There is another way to view formula. Suppose we make the change of variables $w = f(z)$ and hence $dw = f'(z)dz$.
- Since z traverses the curve γ, w will therefore traverse the curve $f(\gamma)$, the image of γ under f;
- this new curve is parametrized by the function $f(\gamma(t))$, where the time parameter t ranges over the same interval as with the original curve.

- By the change of variables formula (which is just as valid for complex integrals as it is for real integrals, and it has the same proof) we thus have

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{f(\gamma)}\frac{dw}{w}$$

- But the right-hand side is nothing more than the winding number of $f(\gamma)$ around *zero*.We have thus proven.

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Theorem (Argument Principle)

Let D be a simply connected domain, and let f be a meromorphic function in D with finitely many zeroes and no removable singularities.

Then for any simple closed curve γ traversed once anti-clockwise which avoids all the poles and zeroes of f, the number of zeroes of f inside γ (counting multiplicity), minus the number of poles of f (counting multiplicity), is equal; to the winding number of $f(\gamma)$ around the origin.

Example

- Consider the function $f(z) := z^5$, and let γ be the curve $\gamma(t) = e^{it} : 0 \leq t \leq 2\pi$.
- Then f has a quintuple zero at 0, and thus has five zeros and no poles inside γ.
- Meanwhile, $f(\gamma)$ is the curve $f(\gamma(t)) = e^{5it} : 0 \leq t \leq 2\pi$, which winds five times anti-clockwise around the origin.
- Since $5 - 0 = 5$, we see that this is consistent with the argument principle.

Now suppose we replace f by the function $g(z) = z^{-5}$;

- Which has five poles inside γ and no zeros.
- Now $g(\gamma)$ is the curve $g(\gamma(t)) = e^{-5it} : 0 \leq t \leq 2\pi$,
- Which winds five times clockwise around the origin.
- Since $0 - 5 = -5$, this is again consistent with the argument principle.

Summery

Informally, what the argument principle is telling us is that every zero of a meromorphic function f twists the complex plane around the origin once anti-clockwise, there as every pole twists the complex plane around the once clockwise.

The total twisting around the origin of a curve γ by the function f (i.e. the winding number of $f(\gamma)$ around the origin) is thus equal to the number of zeroes inside the γ minus the number of poles.

A double zero will twist twice as much as a simple zero, and forth.

End!