MAT 122 2.0 Calculus

Dr. G.H.J. Lanel

Lecture 8

Outline

1. Limits involving infinity
2. The precise $(\epsilon - \delta)$ definition of Infinite Limit
3. Continuous and discontinuous functions

Outline

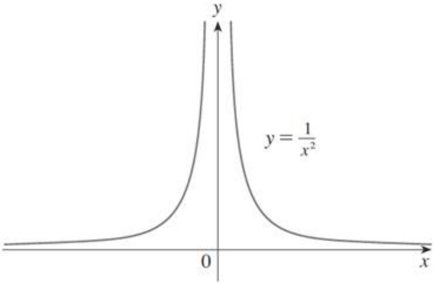
Infinite Limits

Consider the following example:

Find $\lim_{x \to 0} 1/x^2$ if it exists.

Solution :

As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large.



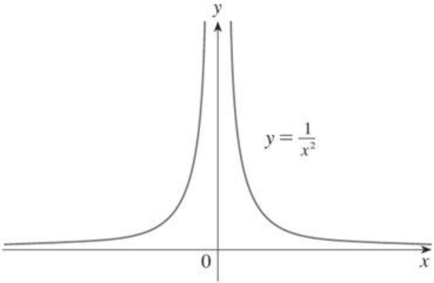
Infinite Limits

Consider the following example:

Find $\lim_{x\to 0} 1/x^2$ if it exists.

Solution :

As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large.



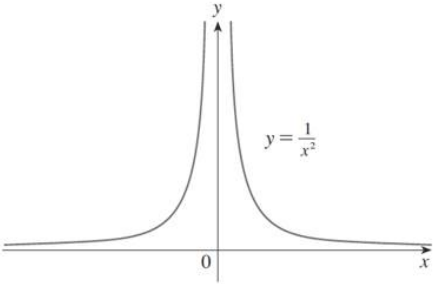
Infinite Limits

Consider the following example:

Find $\lim_{x\to 0} 1/x^2$ if it exists.

Solution :

As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large.



Values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus, the values of $f(x)$ do not approach a number, so $\lim_{x\to 0}\frac{1}{x^2}$ does not exist.

Definition

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a , but not equal to a.

A similar definition can be given for the limit of $f(x)$, as x approaches a in negative infinity.

Similar definitions can be given for the one-sided infinite limits.

$$\lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty$$

Definition

The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = \infty$,
$\lim_{x\to a^-} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^+} f(x) = -\infty$

For instance, the y-axis is a vertical asymptote of the curve $y = 1/x^2$ because $\lim_{x\to 0}(1/x^2) = \infty$

Example:

Find the vertical asymptotes of $f(x) = tanx$

Solution:

$$tanx = \frac{sinx}{cosx}$$

- There are potential vertical asymptotes where $cosx = 0$
- $cosx \to 0^+$ as $x \to (\pi/2)^-$ and $cosx \to 0^-$ as $x \to (\pi/2)^+$
- When $sinx$ is positive when x is near $\pi/2$, we have $\lim_{x\to(\pi/2)^-}(tanx) = \infty$ and $\lim_{x\to(\pi/2)^+}(tanx) = -\infty$
- This shows that the line $\pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = (2n+1)\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = tanx$.

Example:

Find the vertical asymptotes of $f(x) = tanx$

Solution:

$$tanx = \frac{sinx}{cosx}$$

- There are potential vertical asymptotes where $cosx = 0$
- $cosx \to 0^+$ as $x \to (\pi/2)^-$ and $cosx \to 0^-$ as $x \to (\pi/2)^+$
- When $sinx$ is positive when x is near $\pi/2$, we have $\lim_{x\to(\pi/2)^-}(tanx) = \infty$ and $\lim_{x\to(\pi/2)^+}(tanx) = -\infty$
- This shows that the line $\pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = (2n+1)\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = tanx$.

Example:

Find the vertical asymptotes of $f(x) = tanx$

Solution:

$$tanx = \frac{sinx}{cosx}$$

- There are potential vertical asymptotes where $cosx = 0$
- $cosx \to 0^+$ as $x \to (\pi/2)^-$ and $cosx \to 0^-$ as $x \to (\pi/2)^+$
- When $sinx$ is positive when x is near $\pi/2$, we have $\lim_{x\to(\pi/2)^-}(tanx) = \infty$ and $\lim_{x\to(\pi/2)^+}(tanx) = -\infty$
- This shows that the line $\pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = (2n+1)\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = tanx$.

Example:

Find the vertical asymptotes of $f(x) = tanx$

Solution:

$$tanx = \frac{sinx}{cosx}$$

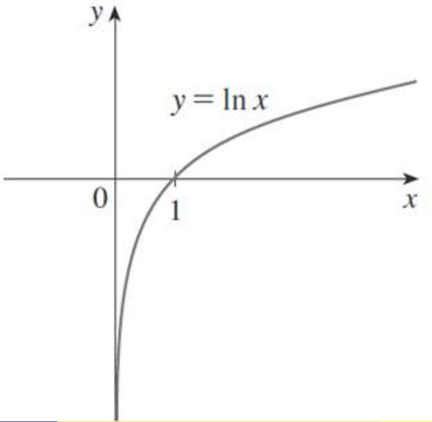
- There are potential vertical asymptotes where $cosx = 0$
- $cosx \to 0^+$ as $x \to (\pi/2)^-$ and $cosx \to 0^-$ as $x \to (\pi/2)^+$
- When $sinx$ is positive when x is near $\pi/2$, we have $\lim_{x\to(\pi/2)^-}(tanx) = \infty$ and $\lim_{x\to(\pi/2)^+}(tanx) = -\infty$
- This shows that the line $\pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = (2n+1)\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = tanx$.

Example: Another example of a function whose graph has a vertical asymptote is the natural logarithmic function $y = lnx$

Solution:

$$\lim_{x \to 0^+} (lnx) = -\infty$$

The line $x = 0$ is a vertical asymptote.

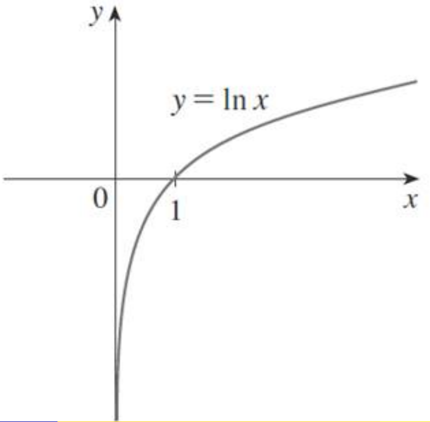


Example: Another example of a function whose graph has a vertical asymptote is the natural logarithmic function $y = lnx$

Solution:

$$\lim_{x \to 0^+} (lnx) = -\infty$$

The line $x = 0$ is a vertical asymptote.

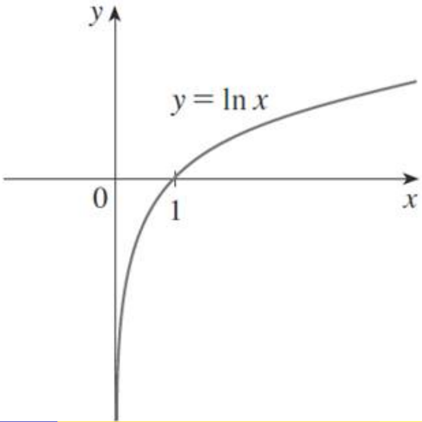


Example: Another example of a function whose graph has a vertical asymptote is the natural logarithmic function $y = lnx$

Solution:

$$\lim_{x \to 0^+} (lnx) = -\infty$$

The line $x = 0$ is a vertical asymptote.

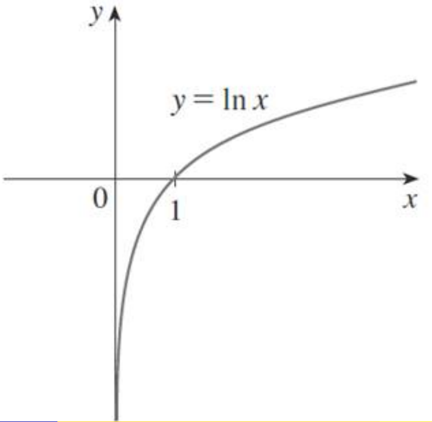


Example: Another example of a function whose graph has a vertical asymptote is the natural logarithmic function $y = lnx$

Solution:

$$\lim_{x \to 0^+} (lnx) = -\infty$$

The line $x = 0$ is a vertical asymptote.



Outline

Definition

Let f be a function defined on some open interval that contains the number a, except possibly at itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$f(x) > M, \text{ whenever } 0 < x - a < \delta$$

Definition

Let f be a function defined on some open interval that contains the number a, except possibly at itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$f(x) > M, \text{ whenever } 0 < x - a < \delta$$

Examples

Example:

Use $\epsilon - \delta$ definition to prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Solution:

1. Guessing a value for δ. Given $M > 0$, we want to find $\delta > 0$ such that $\frac{1}{x^2} > M$, whenever $0 < x - 0 < \delta$

Examples

Example:

Use $\epsilon - \delta$ definition to prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$

Solution:

1. Guessing a value for δ. Given $M > 0$, we want to find $\delta > 0$ such that $\frac{1}{x^2} > M$, whenever $0 < x - 0 < \delta$

Examples

Example:

Use $\epsilon - \delta$ definition to prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Solution:

1. Guessing a value for δ. Given $M > 0$, we want to find $\delta > 0$ such that $\frac{1}{x^2} > M$, whenever $0 < x - 0 < \delta$

that is, $x^2 < \frac{1}{M}$, whenever $0 < x < \delta$, i.e. $x < \frac{1}{\sqrt{M}}$, whenever $0 < x < \delta$

This suggests that we should take $\delta = \frac{1}{\sqrt{M}}$

Showing that this δ works. If $M > 0$ is given, let $\delta = 1/\sqrt{M}$. If $0 < x - 0 < \delta$ then,

$$x < \delta \Rightarrow x^2 < \delta^2$$

$$\Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Thus, $\frac{1}{x^2} > M$, whenever $0 < x - 0 < \delta$

Therefore by definition $\lim_{x \to 0} \frac{1}{x^2} = \infty$

that is, $x^2 < \frac{1}{M}$, whenever $0 < x < \delta$, i.e. $x < \frac{1}{\sqrt{M}}$, whenever $0 < x < \delta$

This suggests that we should take $\delta = \frac{1}{\sqrt{M}}$

Showing that this δ works. If $M > 0$ is given, let $\delta = 1/\sqrt{M}$. If $0 < x - 0 < \delta$ then,

$$x < \delta \Rightarrow x^2 < \delta^2$$
$$\Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Thus, $\frac{1}{x^2} > M$, whenever $0 < x - 0 < \delta$

Therefore by definition $\lim_{x \to 0} \frac{1}{x^2} = \infty$

that is, $x^2 < \frac{1}{M}$, whenever $0 < x < \delta$, i.e. $x < \frac{1}{\sqrt{M}}$, whenever $0 < x < \delta$

This suggests that we should take $\delta = \frac{1}{\sqrt{M}}$

Showing that this δ works. If $M > 0$ is given, let $\delta = 1/\sqrt{M}$. If $0 < x - 0 < \delta$ then,

$$x < \delta \Rightarrow x^2 < \delta^2$$
$$\Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Thus, $\frac{1}{x^2} > M$, whenever $0 < x - 0 < \delta$

Therefore by definition $\lim_{x \to 0} \frac{1}{x^2} = \infty$

that is, $x^2 < \frac{1}{M}$, whenever $0 < x < \delta$, i.e. $x < \frac{1}{\sqrt{M}}$, whenever $0 < x < \delta$

This suggests that we should take $\delta = \frac{1}{\sqrt{M}}$

Showing that this δ works. If $M > 0$ is given, let $\delta = 1/\sqrt{M}$. If $0 < x - 0 < \delta$ then,

$$x < \delta \Rightarrow x^2 < \delta^2$$
$$\Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Thus, $\frac{1}{x^2} > M$, whenever $0 < x - 0 < \delta$

Therefore by definition $\lim_{x \to 0} \frac{1}{x^2} = \infty$

that is, $x^2 < \frac{1}{M}$, whenever $0 < x < \delta$, i.e. $x < \frac{1}{\sqrt{M}}$, whenever $0 < x < \delta$

This suggests that we should take $\delta = \frac{1}{\sqrt{M}}$

Showing that this δ works. If $M > 0$ is given, let $\delta = 1/\sqrt{M}$. If $0 < x - 0 < \delta$ then,

$$x < \delta \Rightarrow x^2 < \delta^2$$
$$\Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} = M$$

Thus, $\frac{1}{x^2} > M$, whenever $0 < x - 0 < \delta$

Therefore by definition $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Outline

1. Limits involving infinity
2. The precise $(\epsilon - \delta)$ definition of Infinite Limit
3. Continuous and discontinuous functions

A function f is continuous at a point $x = a$ if the following are true:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

A function f is continuous at a point $x = a$ if the following are true:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

A function f is continuous at a point $x = a$ if the following are true:

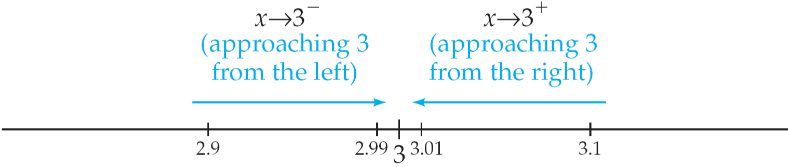
1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

A function f is continuous at a point $x = a$ if the following are true:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

A function f is continuous at a point $x = a$ if the following are true:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$



Examples:

Where are each of the following functions discontinuous?

1. $$f(x) = \frac{x^2 - x - 2}{x - 2}$$

2. $$f(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

Examples:

Where are each of the following functions discontinuous?

1. $$f(x) = \frac{x^2 - x - 2}{x - 2}$$

2. $$f(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

Examples:

Where are each of the following functions discontinuous?

1. $$f(x) = \frac{x^2 - x - 2}{x - 2}$$

2. $$f(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

Examples:

Where are each of the following functions discontinuous?

1. $$f(x) = \frac{x^2 - x - 2}{x - 2}$$

2. $$f(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

Examples Cont...

A function f(x) is defined as follows :

$$f(x) = \begin{cases} 3 + 2x, & \text{if } -\frac{3}{2} < x \leq 0 \\ 3 - 2x, & \text{if } 0 < x \leq \frac{3}{2} \\ -3 - 2x, & \text{if } x > \frac{3}{2} \end{cases}$$

Show that $f(x)$ is continuous at $x = 0$ and is discontinuous at $x = \frac{3}{2}$.

Examples Cont...

A function f(x) is defined as follows :

$$f(x) = \begin{cases} 3 + 2x, & \text{if } -\frac{3}{2} < x \leq 0 \\ 3 - 2x, & \text{if } 0 < x \leq \frac{3}{2} \\ -3 - 2x, & \text{if } x > \frac{3}{2} \end{cases}$$

Show that $f(x)$ is continuous at $x = 0$ and is discontinuous at $x = \frac{3}{2}$.

Examples Cont...

A function f(x) is defined as follows :

$$f(x) = \begin{cases} 3 + 2x, & \text{if } -\frac{3}{2} < x \leq 0 \\ 3 - 2x, & \text{if } 0 < x \leq \frac{3}{2} \\ -3 - 2x, & \text{if } x > \frac{3}{2} \end{cases}$$

Show that $f(x)$ is continuous at $x = 0$ and is discontinuous at $x = \frac{3}{2}$.

Examples Cont...

A function f(x) is defined as follows :

$$f(x) = \begin{cases} 3 + 2x, & \text{if } -\frac{3}{2} < x \leq 0 \\ 3 - 2x, & \text{if } 0 < x \leq \frac{3}{2} \\ -3 - 2x, & \text{if } x > \frac{3}{2} \end{cases}$$

Show that $f(x)$ is continuous at $x = 0$ and is discontinuous at $x = \frac{3}{2}$.

Definition

A function f is continuous from the right at the number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition

A function f is continuous from the right at the number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition

A function f is continuous from the right at the number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition

A function f is continuous from the right at the number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition

A function f is continuous on an interval if it is continuous at every number in the interval.

Note that if f is defined only on one side of an endpoint of the interval, that means the function continuous from the right or continuous from the left.

Eg. 02

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1,1].

Sol: If $-1 < a < 1$, then using the limit laws, we have

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} \left(1 - \sqrt{1 - x^2}\right) \\
&= 1 - \lim_{x\to a} \sqrt{1 - x^2} \\
&= 1 - \sqrt{\lim_{x\to a} (1 - x^2)} \\
&= 1 - \sqrt{1 - a^2} \\
&= f(a)
\end{aligned}$$

Eg. 02
Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1,1].

Sol: If $-1 < a < 1$, then using the limit laws, we have

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left(1 - \sqrt{1 - x^2}\right)\\ &= 1 - \lim_{x\to a}\sqrt{1 - x^2}\\ &= 1 - \sqrt{\lim_{x\to a}(1 - x^2)}\\ &= 1 - \sqrt{1 - a^2}\\ &= f(a)\end{aligned}$$

Eg. 02

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1,1].

Sol: If $-1 < a < 1$, then using the limit laws, we have

$$\begin{aligned}\lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) \\ &= 1 - \lim_{x \to a} \sqrt{1 - x^2} \\ &= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} \\ &= 1 - \sqrt{1 - a^2} \\ &= f(a)\end{aligned}$$

Eg. 02

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1,1].

Sol: If $-1 < a < 1$, then using the limit laws, we have

$$\begin{aligned}\lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) \\ &= 1 - \lim_{x \to a} \sqrt{1 - x^2} \\ &= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} \\ &= 1 - \sqrt{1 - a^2} \\ &= f(a)\end{aligned}$$

Eg. 02

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1,1].

Sol: If $-1 < a < 1$, then using the limit laws, we have

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left(1 - \sqrt{1 - x^2}\right)\\ &= 1 - \lim_{x\to a}\sqrt{1 - x^2}\\ &= 1 - \sqrt{\lim_{x\to a}(1 - x^2)}\\ &= 1 - \sqrt{1 - a^2}\\ &= f(a)\end{aligned}$$

Eg. 02

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1,1].

Sol: If $-1 < a < 1$, then using the limit laws, we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) \\
&= 1 - \lim_{x \to a} \sqrt{1 - x^2} \\
&= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} \\
&= 1 - \sqrt{1 - a^2} \\
&= f(a)
\end{aligned}$$

Eg. 02
Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1,1].

Sol: If $-1 < a < 1$, then using the limit laws, we have

$$\begin{aligned}\lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) \\ &= 1 - \lim_{x \to a} \sqrt{1 - x^2} \\ &= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} \\ &= 1 - \sqrt{1 - a^2} \\ &= f(a)\end{aligned}$$

Eg. 02
Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1,1].

Sol: If $-1 < a < 1$, then using the limit laws, we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) \\
&= 1 - \lim_{x \to a} \sqrt{1 - x^2} \\
&= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} \\
&= 1 - \sqrt{1 - a^2} \\
&= f(a)
\end{aligned}$$

- Thus, by first definition, f is continuous at a if $-1 < a < 1$. Similar calculations show that,

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \text{ and } \lim_{x \to 1^-} f(x) = 1 = f(1)$$

- So f is continuous from the right at -1 and continuous from the left at 1.
- Therefore, according to second definition, f is continuous on [-1,1].

- Thus, by first definition, f is continuous at a if $-1 < a < 1$. Similar calculations show that,

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \text{ and } \lim_{x \to 1^-} f(x) = 1 = f(1)$$

- So f is continuous from the right at -1 and continuous from the left at 1.
- Therefore, according to second definition, f is continuous on [-1,1].

- Thus, by first definition, f is continuous at a if $-1 < a < 1$. Similar calculations show that,

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \text{ and } \lim_{x \to 1^-} f(x) = 1 = f(1)$$

- So f is continuous from the right at -1 and continuous from the left at 1.
- Therefore, according to second definition, f is continuous on [-1,1].

- Thus, by first definition, f is continuous at a if $-1 < a < 1$. Similar calculations show that,

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \text{ and } \lim_{x \to 1^-} f(x) = 1 = f(1)$$

- So f is continuous from the right at -1 and continuous from the left at 1.
- Therefore, according to second definition, f is continuous on [-1,1].

Theorem

If f and g are continuous at a and c is constant, then the following functions are also continuous at a:

1) $f + g$ 2) $f - g$ 3) cf 4) fg 5) $\frac{f}{g}$, if $g(a) \neq 0$

Theorem

If f and g are continuous at a and c is constant, then the following functions are also continuous at a:

1) $f + g$ 2) $f - g$ 3) cf 4) fg 5) $\frac{f}{g}$, if $g(a) \neq 0$

Theorem

If f and g are continuous at a and c is constant, then the following functions are also continuous at a:

1) $f+g$ 2) $f-g$ 3) cf 4) fg 5) $\frac{f}{g}$, if $g(a) \neq 0$

Proof :

Each of the five parts of this theorem follows from the corresponding limit law. For instance, we give the proof of part 1). Since f and g are continuous at a, we have

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

$$\begin{aligned} \therefore \lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ &= f(a)+g(a) \\ &= (f+g)(a) \end{aligned}$$

This shows that $f+g$ is continuous at a.

Proof :

Each of the five parts of this theorem follows from the corresponding limit law. For instance, we give the proof of part 1). Since f and g are continuous at a, we have

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

$$\begin{aligned}\therefore \lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)]\\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)\\ &= f(a)+g(a)\\ &= (f+g)(a)\end{aligned}$$

This shows that $f+g$ is continuous at a.

Proof :

Each of the five parts of this theorem follows from the corresponding limit law. For instance, we give the proof of part 1). Since f and g are continuous at a, we have

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

$$\begin{aligned}\therefore \lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)]\\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)\\ &= f(a)+g(a)\\ &= (f+g)(a)\end{aligned}$$

This shows that $f+g$ is continuous at a.

Proof :

Each of the five parts of this theorem follows from the corresponding limit law. For instance, we give the proof of part 1). Since f and g are continuous at a, we have

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

$$\begin{aligned}\therefore \lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ &= f(a)+g(a) \\ &= (f+g)(a)\end{aligned}$$

This shows that $f+g$ is continuous at a.

Proof :

Each of the five parts of this theorem follows from the corresponding limit law. For instance, we give the proof of part 1). Since f and g are continuous at a, we have

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

$$\begin{aligned}\therefore \lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ &= f(a)+g(a) \\ &= (f+g)(a)\end{aligned}$$

This shows that $f+g$ is continuous at a.

Proof :

Each of the five parts of this theorem follows from the corresponding limit law. For instance, we give the proof of part 1). Since f and g are continuous at a, we have

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

$$\begin{aligned}\therefore \lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ &= f(a)+g(a) \\ &= (f+g)(a)\end{aligned}$$

This shows that $f+g$ is continuous at a.

Proof :

Each of the five parts of this theorem follows from the corresponding limit law. For instance, we give the proof of part 1). Since f and g are continuous at a, we have

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

$$\begin{aligned} \therefore \lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ &= f(a)+g(a) \\ &= (f+g)(a) \end{aligned}$$

This shows that $f+g$ is continuous at a.

Proof :

Each of the five parts of this theorem follows from the corresponding limit law. For instance, we give the proof of part 1). Since f and g are continuous at a, we have

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

$$\begin{aligned}\therefore \lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ &= f(a)+g(a) \\ &= (f+g)(a)\end{aligned}$$

This shows that $f+g$ is continuous at a.

Proof :

Each of the five parts of this theorem follows from the corresponding limit law. For instance, we give the proof of part 1). Since f and g are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a)$$

$$\begin{aligned} \therefore \lim_{x \to a}(f+g)(x) &= \lim_{x \to a}[f(x)+g(x)] \\ &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \\ &= f(a)+g(a) \\ &= (f+g)(a) \end{aligned}$$

This shows that $f+g$ is continuous at a.

Theorem

1. *Any polynomial is continuous everywhere, that is, it is continuous on* $\mathbb{R} = (-\infty, \infty)$.
2. *Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.*

Theorem

1. *Any polynomial is continuous everywhere, that is, it is continuous on* $\mathbb{R} = (-\infty, \infty)$.
2. *Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.*

Theorem

1. *Any polynomial is continuous everywhere, that is, it is continuous on* $\mathbb{R} = (-\infty, \infty)$.
2. *Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.*

Proof:

Proof of Part (1):

A polynomial is the function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \cdots, c_n$ are constants.

We know that

$$\lim_{x \to a} c_0 = c_0$$

and

$$\lim_{x \to a} x^m = a^m, \quad m = 1, 2, \cdots, n$$

Proof:

Proof of Part (1):

A polynomial is the function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \cdots, c_n$ are constants.

We know that

$$\lim_{x \to a} c_0 = c_0$$

and

$$\lim_{x \to a} x^m = a^m, \quad m = 1, 2, \cdots, n$$

Proof:

Proof of Part (1):

A polynomial is the function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \cdots, c_n$ are constants.

We know that

$$\lim_{x \to a} c_0 = c_0$$

and

$$\lim_{x \to a} x^m = a^m, \qquad m = 1, 2, \cdots, n$$

Proof:

Proof of Part (1):

A polynomial is the function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \cdots, c_n$ are constants.

We know that

$$\lim_{x \to a} c_0 = c_0$$

and

$$\lim_{x \to a} x^m = a^m, \quad m = 1, 2, \cdots, n$$

Proof:

Proof of Part (1):

A polynomial is the function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \cdots, c_n$ are constants.

We know that

$$\lim_{x \to a} c_0 = c_0$$

and

$$\lim_{x \to a} x^m = a^m, \qquad m = 1, 2, \cdots, n$$

Proof:

Proof of Part (1):

A polynomial is the function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \cdots, c_n$ are constants.

We know that

$$\lim_{x \to a} c_0 = c_0$$

and

$$\lim_{x \to a} x^m = a^m, \quad m = 1, 2, \cdots, n$$

Proof Contd..

- This equation is precisely the statement that the function $f(x) = x^n$ is a continuous function.
- Thus by part 3) of previous theorem, the function $g(x) = cx^n$ is continuous.
- Since P is a sum of functions of the form and a constant function, it follows from part 1) of previous theorem that P is continuous.

Proof Contd..

Proof of Part (2):

A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

- The domain of f is $D = \{x \in \mathbb{R} | Q(x) \neq 0\}$.
- We know from part (a) that P and Q are continuous everywhere.
- Thus, by part 5) of previous theorem, f is continuous at every number in D.

Proof Contd..

Proof of Part (2):

A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

- The domain of f is $D = \{x \in \mathbb{R} | Q(x) \neq 0\}$.
- We know from part (a) that P and Q are continuous everywhere.
- Thus, by part 5) of previous theorem, f is continuous at every number in D.

Proof Contd..

Proof of Part (2):

A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

- The domain of f is $D = \{x \in \mathbb{R} | Q(x) \neq 0\}$.
- We know from part (a) that P and Q are continuous everywhere.
- Thus, by part 5) of previous theorem, f is continuous at every number in D.

Proof Contd..

Proof of Part (2):

A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

- The domain of f is $D = \{x \in \mathbb{R} | Q(x) \neq 0\}$.
- We know from part (a) that P and Q are continuous everywhere.
- Thus, by part 5) of previous theorem, f is continuous at every number in D.

Proof Contd..

Proof of Part (2):

A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

- The domain of f is $D = \{x \in \mathbb{R} | Q(x) \neq 0\}$.
- We know from part (a) that P and Q are continuous everywhere.
- Thus, by part 5) of previous theorem, f is continuous at every number in D.

Proof Contd..

Proof of Part (2):

A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

- The domain of f is $D = \{x \in \mathbb{R} | Q(x) \neq 0\}$.
- We know from part (a) that P and Q are continuous everywhere.
- Thus, by part 5) of previous theorem, f is continuous at every number in D.

Theorem

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Theorem

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Theorem

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Approaching 3 from the left ↓

x	$2x + 4$
2.9	9.8
2.99	9.98
2.999	9.998

↓ Limit is 10

This table shows $\lim_{x \to 3^-} (2x + 4) = 10$

Approaching 3 from the right ↓

x	$2x + 4$
3.1	10.2
3.01	10.02
3.001	10.002

↓ Limit is 10

This table shows $\lim_{x \to 3^+} (2x + 4) = 10$

Eg.

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$, between 1 and 2.

Sol:

Let $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$. Suppose number c is the root such that $f(c) = 0$.

We take $a = 1, b = 2$ and $N = 0$ in Theorem, we have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$
$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Eg.

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$, between 1 and 2.

Sol:

Let $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$. Suppose number c is the root such that $f(c) = 0$.

We take $a = 1, b = 2$ and $N = 0$ in Theorem, we have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$
$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Eg.

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$, between 1 and 2.

Sol:

Let $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$. Suppose number c is the root such that $f(c) = 0$.

We take $a = 1, b = 2$ and $N = 0$ in Theorem, we have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$
$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Eg.

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$, between 1 and 2.

Sol:

Let $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$. Suppose number c is the root such that $f(c) = 0$.

We take $a = 1, b = 2$ and $N = 0$ in Theorem, we have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$
$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Eg.

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$, between 1 and 2.

Sol:

Let $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$. Suppose number c is the root such that $f(c) = 0$.

We take $a = 1, b = 2$ and $N = 0$ in Theorem, we have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$
$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Eg.

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$, between 1 and 2.

Sol:

Let $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$. Suppose number c is the root such that $f(c) = 0$.

We take $a = 1, b = 2$ and $N = 0$ in Theorem, we have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$
$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

- Thus, $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$.
- Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$.
- In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval (1,2).

- Thus, $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$.
- Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$.
- In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval (1,2).

- Thus, $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$.
- Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$.
- In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval (1,2).