Complex Variables

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Lecture 9

Outline

Note

The argument principle has the following useful consequence:

if one perturbs a complex function by a small amount, then the number of zeroes minus poles that it contains does not change.

Rouche's theorem

Theorem

- *Let γ be a simple closed curve enclosing a domain D, and let f, g be meromorphic functions on $\gamma \cup D$ which have finitely many zeros, no removable singularities, and no poles on γ (i.e. they are all inside γ). Suppose also that*

$$|g(z) - f(z)| < |f(z)| \text{ for all } z \in \gamma;$$

(i.e. at every point z of the curve γ, $g(z)$ is closer to $f(z)$ than the origin is).

- Then $f(\gamma)$ and $g(\gamma)$ have the same winding number around the origin,
- and thus (by the argument principle) the number of zeroes minus poles of $f \in D$ is equal to the number of zeroes minus poles of g.

- This theorem is occasionally called the "Walking the dog" theorem;
- it says that if you (at $f(z)$) are walking a dog (at $g(z)$) around a lamppost (at 0), and you always keep the length of the leash ($|f(z) - g(z)|$) between you and the dog shorter than the distance ($|f(z)|$) to the lamp-post,
- then you and the dog always have the same winding number i.e. the leash cannot get tangled up in the lamp-post.

Outline

Proof

- Let z be any point in γ. By hypothesis we have

  $|f(z)| > |g(z) - f(z)|$, in particular $|f(z)| > 0$, i.e. $f(z)$ is non -zero.

- Also $g(z)$ is non-zero, since if $g(z)$ were zero then $|f(z)|$ would equal $|g(z) - f(z)|$, a contradiction.

- Since if f is non-zero on γ, and we can define the function $h(z)$ by $h(z) := g(z)/f(z)$, which will be analytic everywhere on γ and also meromorphic on D.

- Dividing the hypothesis $|g(z) - f(z)| < |f(z)|$ by $|f(z)|$ we obtain that $|h(z) - 1| < 1$, i.e. for every $z \in \gamma, h(z)$ is contained in the unit ball of radius 1 centered at 1.

- In particular, $h(z)$ is never negative on the real axis, and thus $Logh(z)$ is analytic on γ, where Log is the principal branch of the logarithm.

- This function has derivative $\frac{h'(z)}{h(z)}$ by the chain rule, hence $\frac{h'(z)}{h(z)}$ has anti-derivative $Logh(z)$ on γ.
- Since γ is a closed curve, we thus see from the fundamental theorem of calculus that $\displaystyle\int_\gamma \frac{h'(z)}{h(z)}dz = 0$.

- But recall that $h(z) = \frac{f(z)}{g(z)}$, and hence $\frac{h'(z)}{h(z)} = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}$. Thus

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \int_\gamma \frac{g'(z)}{g(z)} dz$$

- which implies by the argument principle that $f(\gamma)$ and $g(\gamma)$ do have the same winding number around the origin as desired.

Outline

Example

- The function $f(z) := z^5$ has a quintuple zero at the origin. Now consider the function $g(z) := z^5 + z + 1$.
- If we look on the curve $\gamma(t) = 2e^{it} : 0 \leq t \leq 2\pi$, which traverses the circle $z \in \mathbb{C} : |z| = 2$,
- we see that on this curve $|f(z)| = |z^5| = 2^5 = 32$, whereas $|g(z) - f(z)| = |z + 1| \leq |z| + 1 = 2 + 1 = 3$.

Example ctd

- Thus the conditions of Rouche's theorem are satisfied, and the total number of zeroes minus poles of g inside γ must match that of f which is equal to 5.
- Since g is a polynomial, it certainly doesn't have any poles, and hence g has five zeroes inside γ.
- To put it another way, on the curve γ, the term z^5 in g is so much larger than the other two that it dominates the function g.

Example ctd

- And hence by Rouche's theorem g has the same number of zeroes as z^5 inside the curve, i.e. it has five zeroes.
- Indeed we can repeat this analysis for larger circles, i.e.$z \in \mathbb{C} : |z| = R$ for any $R \geq 2$, and conclude that any such circle contains exactly five zeroes of g.
- Thus g has exactly five zeroes in the complex plane, and they all live in the disk $z \in \mathbb{C} : |z| < R$.

End!