AMT 223 1.0 Discrete Mathematics (General Degree)

Dr. G.H.J. Lanel

Semester 2 - 2018

Outline

- We previously showed sequences can be defined recursively.
- Indeed, some sequences have no simple definition other than a recursive one.
- In this section we look at sequences that are not defined recursively (they may be defined in terms of an application) but for which a recursive formula can be written down.

- We previously showed sequences can be defined recursively.
- Indeed, some sequences have no simple definition other than a recursive one.
- In this section we look at sequences that are not defined recursively (they may be defined in terms of an application) but for which a recursive formula can be written down.

- We previously showed sequences can be defined recursively.
- Indeed, some sequences have no simple definition other than a recursive one.
- In this section we look at sequences that are not defined recursively (they may be defined in terms of an application) but for which a recursive formula can be written down.

- Such a formula is called a **recurrence relation** for the sequence.
- The advantage of such a formula is twofold.
- First, it allows us to compute the terms in the sequence, one at a time.
- Second, as we will see in future, it sometimes allows us to derive a closed-form, non-recursive formula for the terms in the sequence.
- Throughout our discussion, the expression defining f can also involve n, even though we do not show it explicitly as an argument.

- Such a formula is called a **recurrence relation** for the sequence.
- The advantage of such a formula is twofold.
- First, it allows us to compute the terms in the sequence, one at a time.
- Second, as we will see in future, it sometimes allows us to derive a closed-form, non-recursive formula for the terms in the sequence.
- Throughout our discussion, the expression defining f can also involve n, even though we do not show it explicitly as an argument.

- Such a formula is called a **recurrence relation** for the sequence.
- The advantage of such a formula is twofold.
- First, it allows us to compute the terms in the sequence, one at a time.
- Second, as we will see in future, it sometimes allows us to derive a closed-form, non-recursive formula for the terms in the sequence.
- Throughout our discussion, the expression defining f can also involve n, even though we do not show it explicitly as an argument.

- Such a formula is called a **recurrence relation** for the sequence.
- The advantage of such a formula is twofold.
- First, it allows us to compute the terms in the sequence, one at a time.
- Second, as we will see in future, it sometimes allows us to derive a closed-form, non-recursive formula for the terms in the sequence.
- Throughout our discussion, the expression defining f can also involve n, even though we do not show it explicitly as an argument.

- Such a formula is called a **recurrence relation** for the sequence.
- The advantage of such a formula is twofold.
- First, it allows us to compute the terms in the sequence, one at a time.
- Second, as we will see in future, it sometimes allows us to derive a closed-form, non-recursive formula for the terms in the sequence.
- Throughout our discussion, the expression defining f can also involve n, even though we do not show it explicitly as an argument.

Examples

- $a_n = na_{n-1}^2 + a_3 + 5$ is a recurrence relation.
- Often only the immediately preceding term of the sequence enters into the recurrence relation, so that $a_n = f(a_{n-1})$ for all $n \geqslant 1$; in this case we say that we have a first-order recurrence relation.
- $a_n = \frac{n^2+1}{a_{n-1}}$ is a first order recurrence relation.

Examples

- $a_n = na_{n-1}^2 + a_3 + 5$ is a recurrence relation.
- Often only the immediately preceding term of the sequence enters into the recurrence relation, so that $a_n = f(a_{n-1})$ for all $n \geqslant 1$; in this case we say that we have a first-order recurrence relation.
- $a_n = \frac{n^2+1}{a_{n-1}}$ is a first order recurrence relation.

Examples

- $a_n = na_{n-1}^2 + a_3 + 5$ is a recurrence relation.
- Often only the immediately preceding term of the sequence enters into the recurrence relation, so that $a_n = f(a_{n-1})$ for all $n \geqslant 1$; in this case we say that we have a first-order recurrence relation.
- $a_n = \frac{n^2+1}{a_{n-1}}$ is a first order recurrence relation.

- More generally, if we have $a_n = f(a_{n-1}, a_{n-2}, ..., a_{n-k})$ for all $n \geqslant k$, then the recurrence relation is said to be of **order** k.
- Often, the restriction $n \geqslant k$ is not written explicitly, but it is to be understood nonetheless, since if $n < k$, then the term a_{n-k} would make no sense.

- More generally, if we have $a_n = f(a_{n-1}, a_{n-2}, ..., a_{n-k})$ for all $n \geqslant k$, then the recurrence relation is said to be of **order** k.
- Often, the restriction $n \geqslant k$ is not written explicitly, but it is to be understood nonetheless, since if $n < k$, then the term a_{n-k} would make no sense.

- Given a sequence that satisfies a kth-order recurrence relation, together with specific values for $a_0, a_1, \ldots, a_{k-1}$, we can write down as many terms of the sequence as we wish.
- The specifications of the values of a_0 through a_{k-1} are called initial conditions.
- Occasionally, the sequence begins at an index other than 0.
- For example,
- we may have a kth-order recurrence relation valid for all $n > k$ with initial conditions specifying $a_1, a_2, \ldots, a_k$

- Given a sequence that satisfies a kth-order recurrence relation, together with specific values for $a_0, a_1, \ldots, a_{k-1}$, we can write down as many terms of the sequence as we wish.
- The specifications of the values of a_0 through a_{k-1} are called initial conditions.
- Occasionally, the sequence begins at an index other than 0.
- For example,
- we may have a kth-order recurrence relation valid for all $n > k$ with initial conditions specifying $a_1, a_2, \ldots, a_k$

- Given a sequence that satisfies a kth-order recurrence relation, together with specific values for $a_0, a_1, \ldots, a_{k-1}$, we can write down as many terms of the sequence as we wish.
- The specifications of the values of a_0 through a_{k-1} are called initial conditions.
- Occasionally, the sequence begins at an index other than 0.
- For example,
- we may have a kth-order recurrence relation valid for all $n > k$ with initial conditions specifying $a_1, a_2, \ldots, a_k$

- Given a sequence that satisfies a kth-order recurrence relation, together with specific values for a_0,a_1, ... ,a_{k-1}, we can write down as many terms of the sequence as we wish.
- The specifications of the values of a_0 through a_{k-1} are called initial conditions.
- Occasionally, the sequence begins at an index other than 0.
- For example,
- we may have a kth-order recurrence relation valid for all $n > k$ with initial conditions specifying a_1,a_2, ... ,a_k

- Given a sequence that satisfies a kth-order recurrence relation, together with specific values for $a_0, a_1, \ldots, a_{k-1}$, we can write down as many terms of the sequence as we wish.
- The specifications of the values of a_0 through a_{k-1} are called initial conditions.
- Occasionally, the sequence begins at an index other than 0.
- For example,
- we may have a kth-order recurrence relation valid for all $n > k$ with initial conditions specifying $a_1, a_2, \ldots, a_k$

Example 1

- We saw in a previous section that the Fibonacci sequence f_n satisfies the second-order recurrence relation $f_n = f_{n-1} + f_{n-2}$ for all $n \geqslant 2$, together with the initial conditions $f_0 = 1$ and $f_1 = 1$.
- Indeed, in this case the recurrence relation and initial conditions form the definition of the sequence.
- Knowing the initial conditions and recurrence relation, we can compute the terms of the sequence, one by one.
- In this case, we find that

$$f_2 = f_1 + f_0 = 1 + 1 = 2$$

$$f_3 = f_2 + f_1 = 2 + 1 = 3$$

$$f_4 = f_3 + f_2 = 3 + 2 = 5$$

and so on.

Example 1

- We saw in a previous section that the Fibonacci sequence f_n satisfies the second-order recurrence relation $f_n = f_{n-1} + f_{n-2}$ for all $n \geqslant 2$, together with the initial conditions $f_0 = 1$ and $f_1 = 1$.
- Indeed, in this case the recurrence relation and initial conditions form the definition of the sequence.
- Knowing the initial conditions and recurrence relation, we can compute the terms of the sequence, one by one.
- In this case, we find that

$$f_2 = f_1 + f_0 = 1 + 1 = 2$$

$$f_3 = f_2 + f_1 = 2 + 1 = 3$$

$$f_4 = f_3 + f_2 = 3 + 2 = 5$$

and so on.

Example 1

- We saw in a previous section that the Fibonacci sequence f_n satisfies the second-order recurrence relation $f_n = f_{n-1} + f_{n-2}$ for all $n \geqslant 2$, together with the initial conditions $f_0 = 1$ and $f_1 = 1$.
- Indeed, in this case the recurrence relation and initial conditions form the definition of the sequence.
- Knowing the initial conditions and recurrence relation, we can compute the terms of the sequence, one by one.
- In this case, we find that

$$f_2 = f_1 + f_0 = 1 + 1 = 2$$

$$f_3 = f_2 + f_1 = 2 + 1 = 3$$

$$f_4 = f_3 + f_2 = 3 + 2 = 5$$

and so on.

Example 1

- We saw in a previous section that the Fibonacci sequence f_n satisfies the second-order recurrence relation $f_n = f_{n-1} + f_{n-2}$ for all $n \geqslant 2$, together with the initial conditions $f_0 = 1$ and $f_1 = 1$.
- Indeed, in this case the recurrence relation and initial conditions form the definition of the sequence.
- Knowing the initial conditions and recurrence relation, we can compute the terms of the sequence, one by one.
- In this case, we find that

$$f_2 = f_1 + f_0 = 1 + 1 = 2$$
$$f_3 = f_2 + f_1 = 2 + 1 = 3$$
$$f_4 = f_3 + f_2 = 3 + 2 = 5$$

and so on.

Example 2

- The sequence given by the explicit formula $a_n = n(n-1)/2$ satisfies the first-order recurrence relation $a_n = a_{n-1} + (n-1)$ for all $n \geqslant 1$, since we have

$$a_{n-1} + (n-1) = \frac{(n-1)(n-2)}{2} + (n-1) = (n-1).\frac{n}{2} = a_n$$

- The initial condition here is that $a_0 = 0$.

Example 2

- The sequence given by the explicit formula $a_n = n(n-1)/2$ satisfies the first-order recurrence relation $a_n = a_{n-1} + (n-1)$ for all $n \geqslant 1$, since we have

$$a_{n-1} + (n-1) = \frac{(n-1)(n-2)}{2} + (n-1) = (n-1).\frac{n}{2} = a_n$$

- The initial condition here is that $a_0 = 0$.

Example 2

- The sequence given by the explicit formula $a_n = n(n-1)/2$ satisfies the first-order recurrence relation $a_n = a_{n-1} + (n-1)$ for all $n \geqslant 1$, since we have

$$a_{n-1} + (n-1) = \frac{(n-1)(n-2)}{2} + (n-1) = (n-1).\frac{n}{2} = a_n$$

- The initial condition here is that $a_0 = 0$.

Example 3

- The sequence 1, 2, 4, 8, 16, ... satisfies the recurrence relation $a_n = a_{n-1} + a_{n-2} + ... + a_1 + a_0 + 1$.
- In other words, each term in this sequence is the sum of all the previous terms, plus 1.
- It also satisfies other recurrence relations, such as the first order relation $a_n = 2a_{n-1}$ and the third order relation $a_n = a_{n-1} + 4a_{n-3}$.

Example 3

- The sequence 1, 2, 4, 8, 16, ... satisfies the recurrence relation $a_n = a_{n-1} + a_{n-2} + ... + a_1 + a_0 + 1$.
- In other words, each term in this sequence is the sum of all the previous terms, plus 1.
- It also satisfies other recurrence relations, such as the first order relation $a_n = 2a_{n-1}$ and the third order relation $a_n = a_{n-1} + 4a_{n-3}$.

Example 3

- The sequence 1, 2, 4, 8, 16, ... satisfies the recurrence relation $a_n = a_{n-1} + a_{n-2} + ... + a_1 + a_0 + 1$.
- In other words, each term in this sequence is the sum of all the previous terms, plus 1.
- It also satisfies other recurrence relations, such as the first order relation $a_n = 2a_{n-1}$ and the third order relation $a_n = a_{n-1} + 4a_{n-3}$.

- Our main goal in this section is to set up recurrence relations for solving problems.
- Typically, we are given a sequence defined in concrete, not algebraic, terms; we want to write down a recurrence relation that the sequence must satisfy.

- Our main goal in this section is to set up recurrence relations for solving problems.
- Typically, we are given a sequence defined in concrete, not algebraic, terms; we want to write down a recurrence relation that the sequence must satisfy.

Example 4

Let b_n be the number of bit strings of length n containing a pair of consecutive 0's. Find a recurrence relation and initial conditions for the sequence b_n.

Solution

- There are three mutually exclusive ways that such a sequence might start: 1, 01, and 00.
- If it starts with a 1, it must continue with a bit string of length $n-1$ containing a pair of consecutive 0's, and there are b_{n-1} of these.
- If it starts with 01, it must continue with a bit string of length $n-2$ containing a pair of consecutive 0's, and there are b_{n-2} of these.

Example 4

Let b_n be the number of bit strings of length n containing a pair of consecutive 0's. Find a recurrence relation and initial conditions for the sequence b_n.

Solution

- There are three mutually exclusive ways that such a sequence might start: 1, 01, and 00.
- If it starts with a 1, it must continue with a bit string of length $n-1$ containing a pair of consecutive 0's, and there are b_{n-1} of these.
- If it starts with 01, it must continue with a bit string of length $n-2$ containing a pair of consecutive 0's, and there are b_{n-2} of these.

Example 4

Let b_n be the number of bit strings of length n containing a pair of consecutive 0's. Find a recurrence relation and initial conditions for the sequence b_n.

Solution

- There are three mutually exclusive ways that such a sequence might start: 1, 01, and 00.
- If it starts with a 1, it must continue with a bit string of length $n-1$ containing a pair of consecutive 0's, and there are b_{n-1} of these.
- If it starts with 01, it must continue with a bit string of length $n-2$ containing a pair of consecutive 0's, and there are b_{n-2} of these.

Example 4

Let b_n be the number of bit strings of length n containing a pair of consecutive 0's. Find a recurrence relation and initial conditions for the sequence b_n.

Solution

- There are three mutually exclusive ways that such a sequence might start: 1, 01, and 00.
- If it starts with a 1, it must continue with a bit string of length $n-1$ containing a pair of consecutive 0's, and there are b_{n-1} of these.
- If it starts with 01, it must continue with a bit string of length $n-2$ containing a pair of consecutive 0's, and there are b_{n-2} of these.

Example 4

Let b_n be the number of bit strings of length n containing a pair of consecutive 0's. Find a recurrence relation and initial conditions for the sequence b_n.

Solution

- There are three mutually exclusive ways that such a sequence might start: 1, 01, and 00.
- If it starts with a 1, it must continue with a bit string of length $n-1$ containing a pair of consecutive 0's, and there are b_{n-1} of these.
- If it starts with 01, it must continue with a bit string of length $n-2$ containing a pair of consecutive 0's, and there are b_{n-2} of these.

Example 4

- Finally, if it starts 00, it can be followed by any bit string of length $n-2$ (since a pair of consecutive 0's is already present), and there are 2^{n-2} of these.
- Therefore, the desired recurrence relation is $b_n = b_{n-1} + b_{n-2} + 2^{n-2}$.
- Clearly, the initial conditions are $b_0 = b_1 = 0$, since no strings of length less than 2 can contain 00 as a substring.

Example 4

- Finally, if it starts 00, it can be followed by any bit string of length $n-2$ (since a pair of consecutive 0's is already present), and there are 2^{n-2} of these.
- Therefore, the desired recurrence relation is $b_n = b_{n-1} + b_{n-2} + 2^{n-2}$.
- Clearly, the initial conditions are $b_0 = b_1 = 0$, since no strings of length less than 2 can contain 00 as a substring.

Example 4

- Finally, if it starts 00, it can be followed by any bit string of length $n-2$ (since a pair of consecutive 0's is already present), and there are 2^{n-2} of these.
- Therefore, the desired recurrence relation is $b_n = b_{n-1} + b_{n-2} + 2^{n-2}$.
- Clearly, the initial conditions are $b_0 = b_1 = 0$, since no strings of length less than 2 can contain 00 as a substring.

Example 4

- With this recurrence relation, we can compute the terms in the sequence.
- We have

$$b_2 = b_1 + b_0 + 2^0 = 0 + 0 + 1 = 1$$

$$b_3 = b_2 + b_1 + 2^1 = 1 + 0 + 2 = 3$$

$$b_4 = b_3 + b_2 + 2^2 = 3 + 1 + 4 = 8$$

$$b_5 = b_4 + b_3 + 2^3 = 8 + 3 + 8 = 19$$

$$b_t6 = b_5 + b_4 + 2^4 = 19 + 8 + 16 = 43$$

$$b_7 = b_6 + b_5 + 2^5 = 43 + 19 + 32 = 94$$

$$b_8 = b_7 + b_6 + 2^6 = 94 + 43 + 64 = 201$$

and so on.

Example 4

- With this recurrence relation, we can compute the terms in the sequence.
- We have

$$b_2 = b_1 + b_0 + 2^0 = 0 + 0 + 1 = 1$$

$$b_3 = b_2 + b_1 + 2^1 = 1 + 0 + 2 = 3$$

$$b_4 = b_3 + b_2 + 2^2 = 3 + 1 + 4 = 8$$

$$b_5 = b_4 + b_3 + 2^3 = 8 + 3 + 8 = 19$$

$$b_t6 = b_5 + b_4 + 2^4 = 19 + 8 + 16 = 43$$

$$b_7 = b_6 + b_5 + 2^5 = 43 + 19 + 32 = 94$$

$$b_8 = b_7 + b_6 + 2^6 = 94 + 43 + 64 = 201$$

and so on.

Example 4

- With this recurrence relation, we can compute the terms in the sequence.
- We have

$$b_2 = b_1 + b_0 + 2^0 = 0 + 0 + 1 = 1$$

$$b_3 = b_2 + b_1 + 2^1 = 1 + 0 + 2 = 3$$

$$b_4 = b_3 + b_2 + 2^2 = 3 + 1 + 4 = 8$$

$$b_5 = b_4 + b_3 + 2^3 = 8 + 3 + 8 = 19$$

$$b_t6 = b_5 + b_4 + 2^4 = 19 + 8 + 16 = 43$$

$$b_7 = b_6 + b_5 + 2^5 = 43 + 19 + 32 = 94$$

$$b_8 = b_7 + b_6 + 2^6 = 94 + 43 + 64 = 201$$

and so on.

Example 4

- With this recurrence relation, we can compute the terms in the sequence.
- We have

$$b_2 = b_1 + b_0 + 2^0 = 0 + 0 + 1 = 1$$

$$b_3 = b_2 + b_1 + 2^1 = 1 + 0 + 2 = 3$$

$$b_4 = b_3 + b_2 + 2^2 = 3 + 1 + 4 = 8$$

$$b_5 = b_4 + b_3 + 2^3 = 8 + 3 + 8 = 19$$

$$b_t6 = b_5 + b_4 + 2^4 = 19 + 8 + 16 = 43$$

$$b_7 = b_6 + b_5 + 2^5 = 43 + 19 + 32 = 94$$

$$b_8 = b_7 + b_6 + 2^6 = 94 + 43 + 64 = 201$$

and so on.

Example 4

- With this recurrence relation, we can compute the terms in the sequence.
- We have

$$b_2 = b_1 + b_0 + 2^0 = 0 + 0 + 1 = 1$$

$$b_3 = b_2 + b_1 + 2^1 = 1 + 0 + 2 = 3$$

$$b_4 = b_3 + b_2 + 2^2 = 3 + 1 + 4 = 8$$

$$b_5 = b_4 + b_3 + 2^3 = 8 + 3 + 8 = 19$$

$$b_t6 = b_5 + b_4 + 2^4 = 19 + 8 + 16 = 43$$

$$b_7 = b_6 + b_5 + 2^5 = 43 + 19 + 32 = 94$$

$$b_8 = b_7 + b_6 + 2^6 = 94 + 43 + 64 = 201$$

and so on.

Example 4

- With this recurrence relation, we can compute the terms in the sequence.
- We have

$$b_2 = b_1 + b_0 + 2^0 = 0 + 0 + 1 = 1$$

$$b_3 = b_2 + b_1 + 2^1 = 1 + 0 + 2 = 3$$

$$b_4 = b_3 + b_2 + 2^2 = 3 + 1 + 4 = 8$$

$$b_5 = b_4 + b_3 + 2^3 = 8 + 3 + 8 = 19$$

$$b_t6 = b_5 + b_4 + 2^4 = 19 + 8 + 16 = 43$$

$$b_7 = b_6 + b_5 + 2^5 = 43 + 19 + 32 = 94$$

$$b_8 = b_7 + b_6 + 2^6 = 94 + 43 + 64 = 201$$

and so on.

Example 4

- With this recurrence relation, we can compute the terms in the sequence.
- We have

$b_2 = b_1 + b_0 + 2^0 = 0 + 0 + 1 = 1$

$b_3 = b_2 + b_1 + 2^1 = 1 + 0 + 2 = 3$

$b_4 = b_3 + b_2 + 2^2 = 3 + 1 + 4 = 8$

$b_5 = b_4 + b_3 + 2^3 = 8 + 3 + 8 = 19$

$b_t 6 = b_5 + b_4 + 2^4 = 19 + 8 + 16 = 43$

$b_7 = b_6 + b_5 + 2^5 = 43 + 19 + 32 = 94$

$b_8 = b_7 + b_6 + 2^6 = 94 + 43 + 64 = 201$

and so on.

Example 4

- With this recurrence relation, we can compute the terms in the sequence.
- We have

$$b_2 = b_1 + b_0 + 2^0 = 0 + 0 + 1 = 1$$

$$b_3 = b_2 + b_1 + 2^1 = 1 + 0 + 2 = 3$$

$$b_4 = b_3 + b_2 + 2^2 = 3 + 1 + 4 = 8$$

$$b_5 = b_4 + b_3 + 2^3 = 8 + 3 + 8 = 19$$

$$b_t6 = b_5 + b_4 + 2^4 = 19 + 8 + 16 = 43$$

$$b_7 = b_6 + b_5 + 2^5 = 43 + 19 + 32 = 94$$

$$b_8 = b_7 + b_6 + 2^6 = 94 + 43 + 64 = 201$$

and so on.

- Unfortunately, there is no algorithm to tell us how to analyze an applied problem, such as the ones we have been considering here, to come up with a recurrence relation.
- A successful analysis often takes a bit of cleverness and usually involves one or more false starts.
- In the rest of this section we turn to problems that are somewhat more involved than the ones we have looked at so far.

- Unfortunately, there is no algorithm to tell us how to analyze an applied problem, such as the ones we have been considering here, to come up with a recurrence relation.
- A successful analysis often takes a bit of cleverness and usually involves one or more false starts.
- In the rest of this section we turn to problems that are somewhat more involved than the ones we have looked at so far.

- Unfortunately, there is no algorithm to tell us how to analyze an applied problem, such as the ones we have been considering here, to come up with a recurrence relation.
- A successful analysis often takes a bit of cleverness and usually involves one or more false starts.
- In the rest of this section we turn to problems that are somewhat more involved than the ones we have looked at so far.

Recurrence Relation not in Closed Form

- Let $p(n)$ be the number of partitions of a set with n elements. ($p(n)$ is also the number of different equivalence relations on a set with n elements.)
- The numbers $p(n)$ are known as the Bell numbers, after the American mathematician E. T. Bell.
- For example, $p(3) = 5$, since the partitions of $\{1, 2, 3\}$ are $\{\{1, 2, 3\}\}$,$\{\{1, 2\}, \{3\}\}$,$\{\{1, 3\}, \{2\}\}$,$\{\{2, 3\}, \{1\}\}$, and $\{\{1\}, \{2\}, \{3\}\}$. In order to get a recurrence relation for $p(n)$, we need to see how partitions of smaller sets help to determine partitions of larger ones.

Recurrence Relation not in Closed Form

- Let $p(n)$ be the number of partitions of a set with n elements. ($p(n)$ is also the number of different equivalence relations on a set with n elements.)
- The numbers $p(n)$ are known as the Bell numbers, after the American mathematician E. T. Bell.
- For example, $p(3) = 5$, since the partitions of $\{1, 2, 3\}$ are $\{\{1, 2, 3\}\}$,$\{\{1, 2\}, \{3\}\}$,$\{\{1, 3\}, \{2\}\}$,$\{\{2, 3\}, \{1\}\}$, and $\{\{1\}, \{2\}, \{3\}\}$. In order to get a recurrence relation for $p(n)$, we need to see how partitions of smaller sets help to determine partitions of larger ones.

Recurrence Relation not in Closed Form

- Let $p(n)$ be the number of partitions of a set with n elements. ($p(n)$ is also the number of different equivalence relations on a set with n elements.)
- The numbers $p(n)$ are known as the Bell numbers, after the American mathematician E. T. Bell.
- For example, $p(3) = 5$, since the partitions of $\{1, 2, 3\}$ are $\{\{1, 2, 3\}\}$,$\{\{1, 2\}, \{3\}\}$,$\{\{1, 3\}, \{2\}\}$,$\{\{2, 3\}, \{1\}\}$, and $\{\{1\}, \{2\}, \{3\}\}$. In order to get a recurrence relation for $p(n)$, we need to see how partitions of smaller sets help to determine partitions of larger ones.

- We count the partitions of $\{1, 2, .., n\}$ as follows.
- The element n must be in one of the sets of the partition.
- It can be in a set by itself, or it can have one or more (possibly even all) of the other elements of $\{1, 2, , n\}$ with it.
- Let k be the number of elements other than n in the same set with n in a partition of $\{1, 2, ..., n\}$.
- For example, if $n = 3$, then the partition $\{\{2, 3\}, \{1\}\}$ has $k = 1$ since only 2 is in the same set as 3.

- We count the partitions of $\{1, 2, .., n\}$ as follows.
- The element n must be in one of the sets of the partition.
- It can be in a set by itself, or it can have one or more (possibly even all) of the other elements of $\{1, 2, , n\}$ with it.
- Let k be the number of elements other than n in the same set with n in a partition of $\{1, 2, ..., n\}$.
- For example, if $n = 3$, then the partition $\{\{2, 3\}, \{1\}\}$ has $k = 1$ since only 2 is in the same set as 3.

- We count the partitions of $\{1, 2, .., n\}$ as follows.
- The element n must be in one of the sets of the partition.
- It can be in a set by itself, or it can have one or more (possibly even all) of the other elements of $\{1, 2, , n\}$ with it.
- Let k be the number of elements other than n in the same set with n in a partition of $\{1, 2, ..., n\}$.
- For example, if $n = 3$, then the partition $\{\{2, 3\}, \{1\}\}$ has $k = 1$ since only 2 is in the same set as 3.

- We count the partitions of $\{1, 2, .., n\}$ as follows.
- The element n must be in one of the sets of the partition.
- It can be in a set by itself, or it can have one or more (possibly even all) of the other elements of $\{1, 2, , n\}$ with it.
- Let k be the number of elements other than n in the same set with n in a partition of $\{1, 2, ..., n\}$.
- For example, if $n = 3$, then the partition $\{\{2, 3\}, \{1\}\}$ has $k = 1$ since only 2 is in the same set as 3.

- We count the partitions of $\{1, 2, .., n\}$ as follows.
- The element n must be in one of the sets of the partition.
- It can be in a set by itself, or it can have one or more (possibly even all) of the other elements of $\{1, 2, , n\}$ with it.
- Let k be the number of elements other than n in the same set with n in a partition of $\{1, 2, ..., n\}$.
- For example, if $n = 3$, then the partition $\{\{2, 3\}, \{1\}\}$ has $k = 1$ since only 2 is in the same set as 3.

- Note that $1 \leqslant k \leqslant n-1$.

- In order to specify a partition with this value of k, we can first decide which k elements are to be in the same set as n (and we can do this in $C(n-1,k)$ ways), and then decide how to partition the remaining elements of $\{1,2,...,n\}$ (and we can do this in $p(n-k-1)$ ways, since there are $n-k-1$ elements left to be partitioned).

- Therefore, by the multiplication principle there are $C(n-1,k)p(n-k-1)$ partitions of $\{1,2.,..n\}$ in which exactly k elements are in the same set as n.

- Note that $1 \leqslant k \leqslant n-1$.
- In order to specify a partition with this value of k, we can first decide which k elements are to be in the same set as n (and we can do this in $C(n-1, k)$ ways), and then decide how to partition the remaining elements of $\{1, 2, ..., n\}$ (and we can do this in $p(n-k-1)$ ways, since there are $n-k-1$ elements left to be partitioned).
- Therefore, by the multiplication principle there are $C(n-1, k)p(n-k-1)$ partitions of $\{1, 2., ..n\}$ in which exactly k elements are in the same set as n.

- Note that $1 \leqslant k \leqslant n-1$.
- In order to specify a partition with this value of k, we can first decide which k elements are to be in the same set as n (and we can do this in $C(n-1, k)$ ways), and then decide how to partition the remaining elements of $\{1, 2, ..., n\}$ (and we can do this in $p(n-k-1)$ ways, since there are $n-k-1$ elements left to be partitioned).
- Therefore, by the multiplication principle there are $C(n-1, k)p(n-k-1)$ partitions of $\{1, 2., ..n\}$ in which exactly k elements are in the same set as n.

- Finally, by the addition principle, the total number of partitions of $\{1, 2, ...n\}$ is given by

$$P(n) = \sum_{k=0}^{n-1} C(n-1, k)p(n-k-1).$$

- This formula is our recurrence relation; it specifies $p(n)$ in terms of the numbers $p(n-k-1)$, all of which have arguments smaller than n (since $k \geqslant 0$). The only initial condition needed is $p(0) = 1$, reflecting the fact that the empty set is the only partition of the empty set.

- Note that this recurrence relation is not of a fixed order, as were most of the recurrence relations we considered earlier in this section.

- Instead, the recurrence relation expresses $p(n)$ as a function of all the numbers $p(0)$, $p(1)$,...,$p(n-1)$ (as well as n).

- Finally, by the addition principle, the total number of partitions of $\{1, 2, ...n\}$ is given by

$$P(n) = \sum_{k=0}^{n-1} C(n-1, k)p(n-k-1).$$

- This formula is our recurrence relation; it specifies $p(n)$ in terms of the numbers $p(n-k-1)$, all of which have arguments smaller than n (since $k \geqslant 0$). The only initial condition needed is $p(0) = 1$, reflecting the fact that the empty set is the only partition of the empty set.

- Note that this recurrence relation is not of a fixed order, as were most of the recurrence relations we considered earlier in this section.

- Instead, the recurrence relation expresses $p(n)$ as a function of all the numbers $p(0)$, $p(1)$,...,$p(n-1)$ (as well as n).

- Finally, by the addition principle, the total number of partitions of $\{1, 2, ...n\}$ is given by

$$P(n) = \sum_{k=0}^{n-1} C(n-1, k)p(n-k-1).$$

- This formula is our recurrence relation; it specifies $p(n)$ in terms of the numbers $p(n-k-1)$, all of which have arguments smaller than n (since $k \geqslant 0$). The only initial condition needed is $p(0) = 1$, reflecting the fact that the empty set is the only partition of the empty set.

- Note that this recurrence relation is not of a fixed order, as were most of the recurrence relations we considered earlier in this section.

- Instead, the recurrence relation expresses $p(n)$ as a function of all the numbers $p(0)$, $p(1)$,...,$p(n-1)$ (as well as n).

- Finally, by the addition principle, the total number of partitions of $\{1, 2, ...n\}$ is given by

$$P(n) = \sum_{k=0}^{n-1} C(n-1, k)p(n-k-1).$$

- This formula is our recurrence relation; it specifies $p(n)$ in terms of the numbers $p(n-k-1)$, all of which have arguments smaller than n (since $k \geqslant 0$). The only initial condition needed is $p(0) = 1$, reflecting the fact that the empty set is the only partition of the empty set.

- Note that this recurrence relation is not of a fixed order, as were most of the recurrence relations we considered earlier in this section.

- Instead, the recurrence relation expresses $p(n)$ as a function of all the numbers $p(0)$, $p(1)$,...,$p(n-1)$ (as well as n).

Example

Find the number of partitions of a set with five elements.

Solution

- We need to compute $p(5)$, and we can do so if we first compute $p(0)$, $p(1)$, $p(2)$, $p(3)$, and $p(4)$.
- The first three of these we may as well do directly, since they are so simple.
- We already noted that $p(0) = 1$, and it is clear that $p(1) = 1$ as well.
- Also, $p(2) = 2$, since we can put the two elements either in one set together or in separate sets.
- Similarly $p(3) = 5$ several paragraphs above.

Example

Find the number of partitions of a set with five elements.

Solution

- We need to compute $p(5)$, and we can do so if we first compute $p(0)$, $p(1)$, $p(2)$, $p(3)$, and $p(4)$.
- The first three of these we may as well do directly, since they are so simple.
- We already noted that $p(0) = 1$, and it is clear that $p(1) = 1$ as well.
- Also, $p(2) = 2$, since we can put the two elements either in one set together or in separate sets.
- Similarly $p(3) = 5$ several paragraphs above.

Example

Find the number of partitions of a set with five elements.

Solution

- We need to compute $p(5)$, and we can do so if we first compute $p(0)$, $p(1)$, $p(2)$, $p(3)$, and $p(4)$.
- The first three of these we may as well do directly, since they are so simple.
- We already noted that $p(0) = 1$, and it is clear that $p(1) = 1$ as well.
- Also, $p(2) = 2$, since we can put the two elements either in one set together or in separate sets.
- Similarly $p(3) = 5$ several paragraphs above.

Example

Find the number of partitions of a set with five elements.

Solution

- We need to compute $p(5)$, and we can do so if we first compute $p(0)$, $p(1)$, $p(2)$, $p(3)$, and $p(4)$.
- The first three of these we may as well do directly, since they are so simple.
- We already noted that $p(0) = 1$, and it is clear that $p(1) = 1$ as well.
- Also, $p(2) = 2$, since we can put the two elements either in one set together or in separate sets.
- Similarly $p(3) = 5$ several paragraphs above.

Example

Find the number of partitions of a set with five elements.

Solution

- We need to compute $p(5)$, and we can do so if we first compute $p(0)$, $p(1)$, $p(2)$, $p(3)$, and $p(4)$.
- The first three of these we may as well do directly, since they are so simple.
- We already noted that $p(0) = 1$, and it is clear that $p(1) = 1$ as well.
- Also, $p(2) = 2$, since we can put the two elements either in one set together or in separate sets.
- Similarly $p(3) = 5$ several paragraphs above.

Example

Find the number of partitions of a set with five elements.

Solution

- We need to compute $p(5)$, and we can do so if we first compute $p(0)$, $p(1)$, $p(2)$, $p(3)$, and $p(4)$.
- The first three of these we may as well do directly, since they are so simple.
- We already noted that $p(0) = 1$, and it is clear that $p(1) = 1$ as well.
- Also, $p(2) = 2$, since we can put the two elements either in one set together or in separate sets.
- Similarly $p(3) = 5$ several paragraphs above.

Example

Find the number of partitions of a set with five elements.

Solution

- We need to compute $p(5)$, and we can do so if we first compute $p(0)$, $p(1)$, $p(2)$, $p(3)$, and $p(4)$.
- The first three of these we may as well do directly, since they are so simple.
- We already noted that $p(0) = 1$, and it is clear that $p(1) = 1$ as well.
- Also, $p(2) = 2$, since we can put the two elements either in one set together or in separate sets.
- Similarly $p(3) = 5$ several paragraphs above.

- To compute $p(4)$ we use the recurrence relation:

  $P(4) = \sum_{k=0}^{3} C(3,k)p(3-k)$.

  $= C(3,0)P(3) + C(3,1)P(2) + C(3,2)p(1) + C(3,3)p(0)$

  $= 1 \cdot 5 + 3 \cdot 2 + 3 \cdot 1 + 1 \cdot 1 = 15$

- Finally, we use the recurrence relation again to find $p(5)$:

  $P(5) = \sum_{k=0}^{4} C(4,k)p(4-k)$.

  $= C(4,0)P(4) + C(4,1)P(3) + C(4,2)p(2) + C(4,3)p(1) + C(4,4)p(0)$

  $= 1 \cdot 15 + 4 \cdot 5 + 6 \cdot 2 + 4 \cdot 1 + 1 \cdot 1 = 52$

- To compute $p(4)$ we use the recurrence relation:

  $P(4) = \sum_{k=0}^{3} C(3,k)p(3-k).$

  $= C(3,0)P(3) + C(3,1)P(2) + C(3,2)p(1) + C(3,3)p(0)$

  $= 1 \cdot 5 + 3 \cdot 2 + 3 \cdot 1 + 1 \cdot 1 = 15$

- Finally, we use the recurrence relation again to find $p(5)$:

  $P(5) = \sum_{k=0}^{4} C(4,k)p(4-k).$

  $=$

  $C(4,0)P(4) + C(4,1)P(3) + C(4,2)p(2) + C(4,3)p(1) + C(4,4)p(0)$

  $= 1 \cdot 15 + 4 \cdot 5 + 6 \cdot 2 + 4 \cdot 1 + 1 \cdot 1 = 52$

- Thus there are exactly 52 partitions of the set (1, 2, 3, 4, 5) (or any other set with five elements).
- It would have been difficult to be sure of obtaining the right answer by trying to list these 52 partitions.

- Thus there are exactly 52 partitions of the set (1, 2, 3, 4, 5) (or any other set with five elements).
- It would have been difficult to be sure of obtaining the right answer by trying to list these 52 partitions.