MAT 122 2.0 Calculus

Dr. G.H.J. Lanel

Lecture 9

Outline

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We considered the derivative of a function f at a fixed number a:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

Here we change our point of view and let the number a vary. If we replace a in equation by a variable x, we obtain

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

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- Given any number x for which this limit exists, we assign to x the number $f'(x)$.
- So we can regard f' as a new function, called the derivative of f and defined by equation (2).
- We know that the value of f' at x, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.
- The function f' is called the derivative of f because it has been derived from f by the limiting operation in equation (2).
- The domain of f' is the set $\{x|f'(x)\ exists\}$ and may be smaller than the domain of f.

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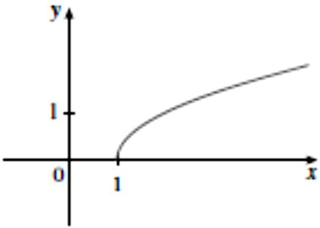
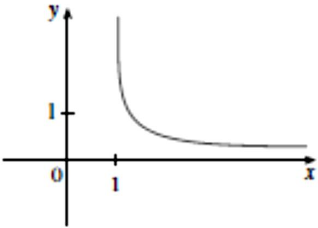
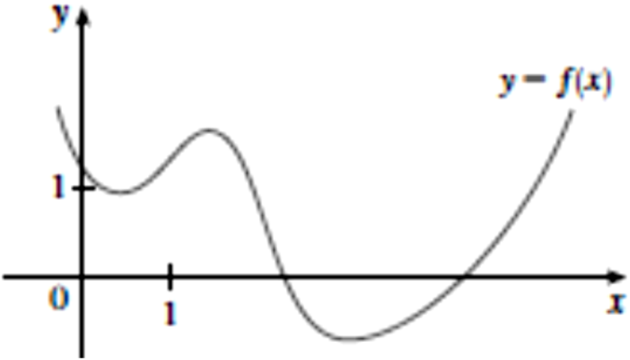


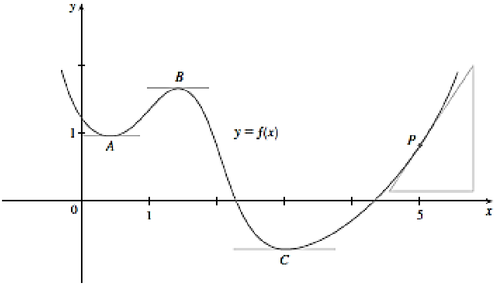
(a) $f(x) = \sqrt{x-1}$



(b) $f'(x) = \dfrac{1}{2\sqrt{x-1}}$

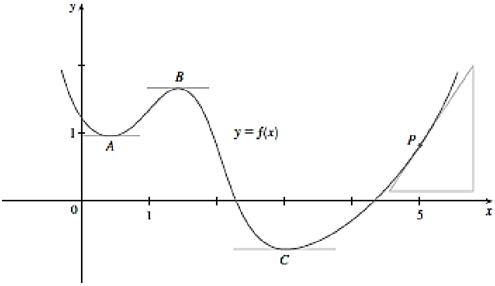


Sketch the graph of the derivative f'.



(a)

- Tangents at A, B and C are horizontal, so the derivative is 0 there, and the graph of f' crosses the x-axis at the points A', B' and C'..
- Between A and B the tangents have positive slope, so $f'(x)$ is positive there.
- But between B and C the tangents have negative slope, so $f'(x)$ is negative there.



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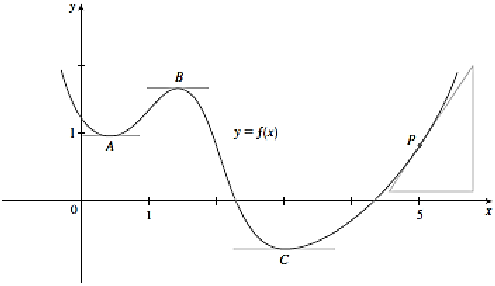

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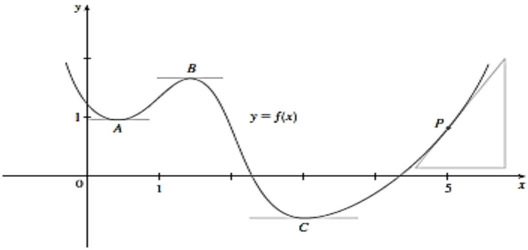

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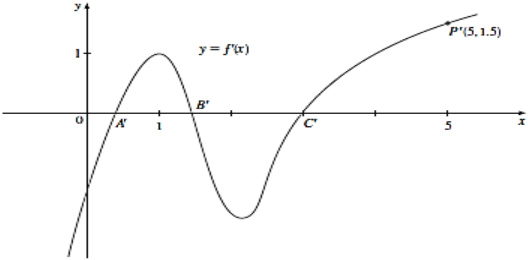

(b)

Outline

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(X) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and $\frac{d}{dx}$ are called differentiable operators because they indicate the operation of differentiation, which is the process of calculating a derivative.

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Definition

A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) [*or* (a, ∞) *or* $(-\infty, a)$ *or* $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Eg. $f(x) = |x|$.

If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$ we have

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\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{|x+h| - |x|}{h} \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h} \\
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$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h} \quad \text{(if it exists)}$$

Let's compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

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Note: The left-hand and right-hand derivatives of f at a are defined by

$$f'_{-}(a) = \lim_{h \to 0, h > 0} \frac{f(a) - f(a-h)}{h}$$

and

$$f'_{+}(a) = \lim_{h \to 0, h > 0} \frac{f(a+h) - f(a)}{h}$$

if there limits exist. Then $f'(a)$ exists iff these one-sided derivatives are exist and equal.

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Ex.

1. Find $f'_-(4)$ and $f'_+(4)$ for the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \frac{1}{5-x} & \text{if } x \geq 4 \end{cases}$$

2. Sketch the graph of f.
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Theorem

If f is differentiable at a, then f is continuous at a.

Proof :

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a); \ (x \neq a)$$

$$\begin{aligned} \lim_{x \to a}[f(x) - f(a)] &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a}(x - a) \\ &= f'(a) \cdot 0 \\ &= 0 \end{aligned}$$

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Note:

For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

But in Example we showed that f is not differentiable at 0.

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Outline

Second derivative

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function $f''$is called the **second derivative** of because it is the derivative of the derivative of f . We write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

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Example:

If $f(x) = x^3 - x$, find an interpret $f''(x)$.

Solution:

first derivative is$f'(x) = 3x^2 - 1$. So the second derivative is;

$f''(x) = (f')'(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0} \frac{[3*(x+h)^2-1]-[3x^2-1]}{h} = \lim_{h\to 0} \frac{3x^2+6xh3h^2-1-3x^2+1}{h} = \lim_{h\to 0} \frac{6x+3h}{h} = 6x$

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In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is acceleration, which we define as follows. If $s = s(t)$is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the acceleration $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

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Third derivative

Third derivative is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$. If $y = f(x)$, then alternative notations for the third derivative are,

$$y''' = f'''(x) = \frac{d}{dx}(\frac{d^2y}{dx^2}) = \frac{d^3y}{dx^3}$$

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

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