Graph Theory and Its Applications

Dr. G.H.J. Lanel

Lecture 11

Outline

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Matching problems

- Some real-world problems involve finding matching pairs in a group.
- For example, we might want to allocate jobs to candidates. There are a number of candidates who are qualified for each job; what is the arrangement which leaves all positions filled?
- What if the candidates are qualified for different jobs to different extents. i.e. some matchings are preferable to others?

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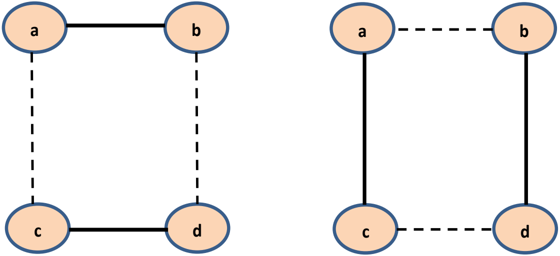
Matchings



Two possible matchings in a simple graph.

Matchings

- A matching is a subset of edges in a graph which have no common vertices.
- For each edge M in a matching, the two vertices at either end are matched.
- A matching M is maximum if as many vertices are matched as possible.
- A perfect matching is one in which every vertex is matched.
- An M-alternating path in a graph is one in which the edges are alternately in M and $G \setminus M$.

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Maximal Matchings

Definition

Given a graph $G = \langle V, E \rangle$ a **matching** is a collection of edges M such that $e_i, e_j \in M \Rightarrow e_i, e_j$ are vertex disjoint.

Definition

A maximal matching is a matching that can not be improved w.r.t. current matching.

Definition

A **maximum matching** is a matching of maximum cardinality (over all possible matchings in a graph G); v(G) denotes the cardinality of a maximum matching in G.

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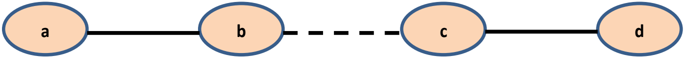

(a maximal but not maximum matching; a maximum matching)

Definition

Given a matching M, an M-**exposed** vertex is a vertex not incident with any edge in M; an M-**covered** vertex is a vertex incident with any edge in M.

Proposition

Given a graph $G = \langle V, E \rangle$, a ***perfect matching*** *is a matching with deficiency $def(G) = |V| - 2 \cdot v(G) = 0$.*

Definition

Given a matching M in a graph G, a path P composed of edges that alternately belong to and do not belong to M is called an **M-alternating path.**

Definition

An M-alternating path P is an M-augmenting path if the first and last vertices are M-exposed.

Theorem

A matching M in a graph $G = \langle V, E \rangle$ is maximum if and only if there is no M-augmenting path.

A matching $M = \{e_{ab}\}$

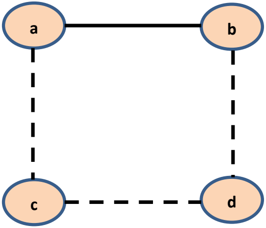

Figure shows a simple graph with a matching $M = \{e_{ab}\}$. The path from c to a to b to d is an M alternating path, because the edges in the path alternately belong to the matching ($e_{ca} \notin M$; $e_{ab} \in M$; $e_{bd} \notin M$). The same path is also an M-augmenting path because its endpoints, c and d are M-exposed; that is, they are not incident with any edge in M.

Outline

Let $G = (V, E)$, be a graph. A subgraph is called a matching $M(G)$, if each vertex of G is incident with at most one edge in M,
i.e. in the matching, $deg(v) \leq 1, \forall v \in G$,

which means in the matching graph $M(G)$, the vertices should have a degree of 1 or 0, where the edges should be incident from the graph G.

Example

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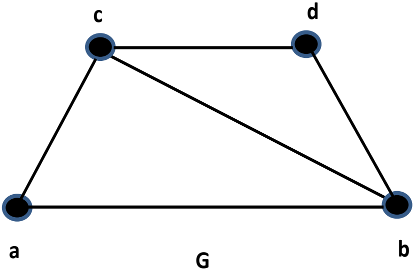

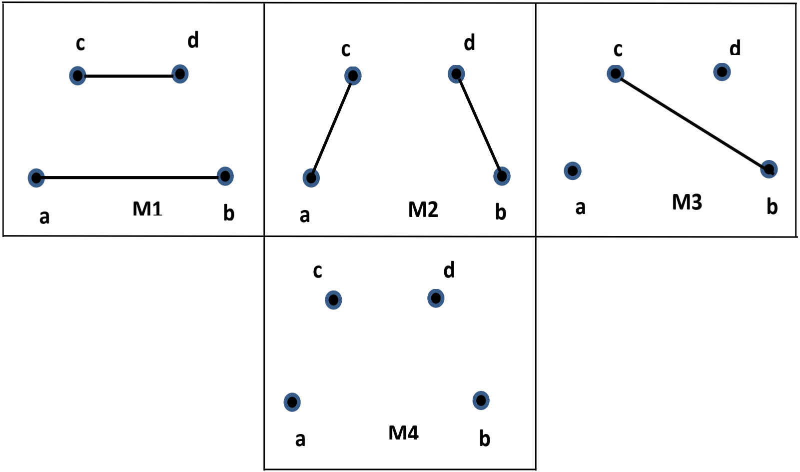
c
d
a
M1
b
c
d
a
M2
b
c
d
a
M3
b
c
d
a
M4
b

M_1, M_2, M_3 from the above graph are the maximal matching of G.

Example of Maximum matching

Maximum matching is defined as the maximal matching with maximum number of edges. It is also known as largest maximal matching. The number of edges in the maximum matching of G is called its matching number.

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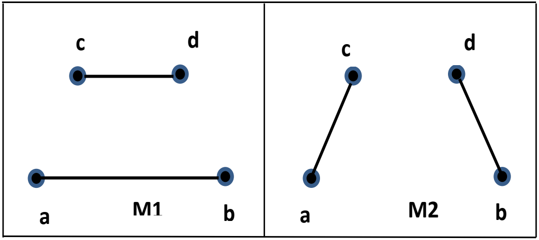

For a graph given in the above example, M_1 and M_2 are the maximum matching of G and its matching number is 2.

Example of Perfect matching

A matching M of graph G is said to be a perfect match, if every vertex of graph G is incident to exactly one edge of the matching M,

i.e. in the matching, $deg(v) = 1, \forall v \in G$

The degree of each and every vertex in the subgraph should have a degree of 1.

M_1 and M_2 are examples of perfect matching of G.

Note: Every perfect matching of graph is also a maximum matching of graph, because there is no chance of adding one more edge in a perfect matching graph.

A maximum matching of graph need not be perfect.

If a graph G has a perfect match, then the number of vertices $|V|$ is even.

If it is odd, then the last vertex pairs with the other vertex, and finally there remains a single vertex which cannot be paired with any other vertex for which the degree is zero. It clearly violates the perfect matching principle.

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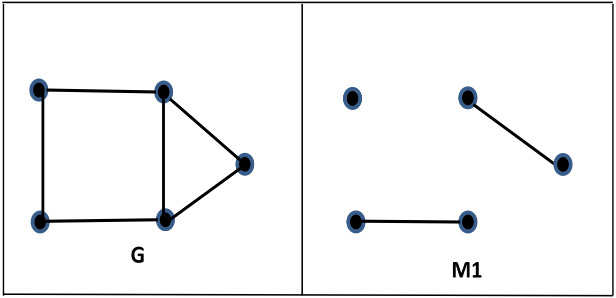

Note: The converse of the above statement need not be true. If G has even number of vertices, then the maximum matching need not be perfect.

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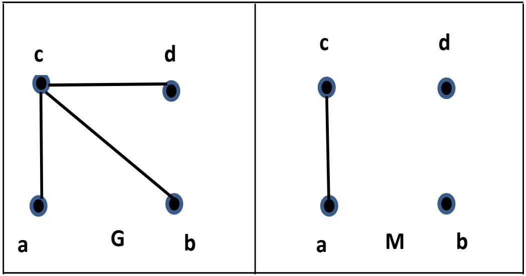

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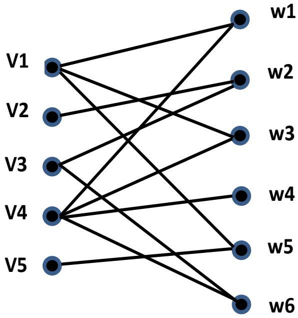
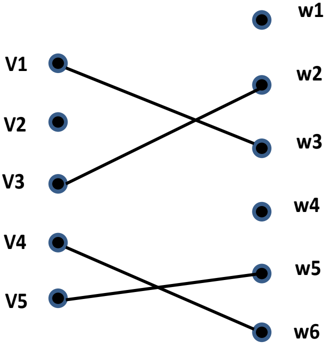
G:



S:



Examples

For the matching $S = \{(v_1, w_3), (v_3, w_2), (v_4, w_6), (v_5, w_5)\}$, an augmenting path (p) is given by the vertices with the order $v_2, w_2, v_3, w_6, v_4, w_1$.

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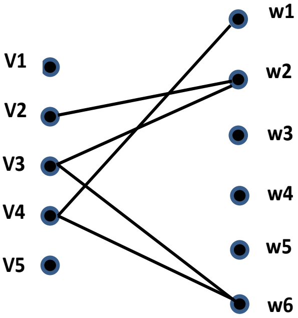

We can augment a matching S using its augmenting path p as follows:

We remove the edges of S in p, and add the edges in p which are not in S.

The new edge set is obviously a matching. Note that the number of edges in S on an augmenting path is one fewer than the number of the remaining edges. Therefore, the number of edges in a matching increases by one after the augmenting operation.

Theorem

In a society of m men and w women, w marriages between women and men they are acquainted with are possible iff each subset of k women $(1 \leq k \leq w)$ is acquainted with at least k men.

Proof

- *The condition is clearly necessary. We prove the sufficiency by strong induction on w.*
- *Induction statement $S(n)$: If in a collection of n women, each subset of k women $(1 \leq k \leq n)$ collectively is acquainted with at least k men, then n marriages are possible.*
- *$S(1)$: Since there is only one woman, and she knows at least one man, "heavenly bliss" is possible.*
- *Now assume $S(i)$ is true for all $i \leq n$. Consider $S(n+1)$. There are two cases to consider:*

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- *Now assume $S(i)$ is true for all $i \leq n$. Consider $S(n+1)$. There are two cases to consider:*

Case 1: Every set of k women ($1 \leq k \leq n$) knows at least $k + 1$ men. In this case, take one woman and a man she is acquainted with and marry them off. Now, there are only n women left, and every subset of k of them collectively know at least k men. By the induction hypothesis, n marriages are possible. Together with the original marriage, we get $n + 1$ marriages.

Case 2: Suppose that there is a set of k women ($1 \leq k \leq n$) who collectively know exactly k men. Since this set of k women and k men satisfy the condition, by induction we can arrange k marriages. There remain $n + 1 - k$ women. Any subset of j of these women must collectively know at least j men, otherwise these j women together with the k women already married would have collectively known less than $k + j$ men (contradicting the assumption for the $n + 1$ women). So, the induction hypothesis is valid for these $n + 1 - k$ women, and we can arrange $n + 1 - k$ marriages. Together with the previous k marriages, we have arranged $n + 1$ marriages.

Thus, by induction, the statement is true for all w.

Outline

Definition

Given a graph $G = \langle V, E \rangle$, a **cover** is a set of vertices $A \subseteq V$ such that for every edge $e = vw \in E$, either $v \in A$ or $w \in A$.

Outline

Example of Line covering

Let $G = (V, E)$, be a graph. A subset $C(E)$ is called a line covering of G if every vertex of G is incident with at least one edge in C,

i.e., $deg(v) \geq 1, \forall v \in G$,

because each vertex is connected with another vertex by an edge. Hence it has a minimum degree of 1.

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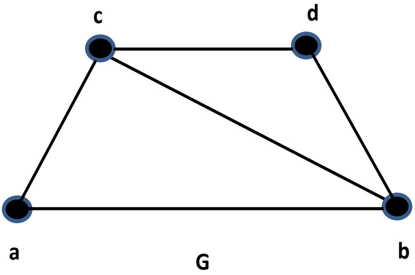


Its subgraphs having line covering are as follows:

$C_1 = \{\{a, b\}, \{c, d\}\}$

$C_2 = \{\{a, c\}, \{b, d\}\}$

$C_3 = \{\{a, b\}, \{b, c\}, \{b, d\}\}$

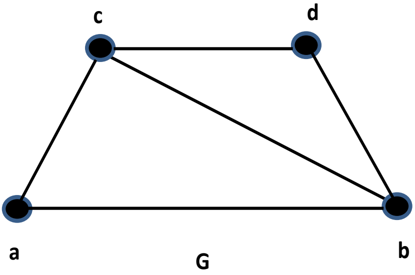
$C_4 = \{\{a, b\}, \{b, c\}, \{c, d\}\}$

Line covering of G does not exist if and only if G has an isolated vertex.

Line covering of a graph with n vertices has at least $[n/2]$ edges.

Examples of Minimal Line covering

A line covering C of a graph G is said to be minimal if no edge can be deleted from C.



In the above graph, C_1, C_2, C_3 are minimal line coverings, while C_4 is not because we can delete $\{b, c\}$.

Examples of Minimum Line covering

It is also known as smallest minimal line covering. A minimal line covering with minimum number of edges is called a minimum line covering of G. The number of edges in a minimum line covering in G is called the line covering number (α).

In the above example, C_1 and C_2 are the minimum line covering of G and $\alpha = 2$.

- Every line covering contains a minimal line covering.
- Every line covering does not contain a minimum line covering (C_3 does not contain any minimum line covering).
- No minimal line covering contains a cycle.
- If a line covering C contains no paths of length 3 or more, then C is a minimal line covering because all the components of C are star graph and from a star graph, no edge can be deleted.

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Examples of Vertex covering

Let $G = (V, E)$, be a graph. A subset K of V is called a vertex covering of G, if every edge of G is incident with or covered by a vertex in K.

The subgraphs that can be derived from the above graph are as follows:

$K_1 = \{b, c\}$

$K_2 = \{a, b, c\}$

$K_3 = \{b, c, d\}$

$K_4 = \{a, d\}$

Here, K_1, K_2, and K_3 have vertex covering, whereas K_4 does not have any vertex covering as it does not cover the edge $\{bc\}$.

Examples of Minimal Vertex covering

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In the above graph, the subgraphs having vertex covering are as follows:

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Here, K_1 and K_2 are minimal vertex coverings, whereas in K_3, vertex d can be deleted.

Examples of Minimum Vertex covering

It is also known as the smallest minimal vertex covering. A minimal vertex covering of graph G with minimum number of vertices is called the minimum vertex covering.

The number of vertices in a minimum vertex covering of G is called the vertex covering number (β).

In the above graph, K_1 is a minimum vertex cover of G, as it has only two vertices. $\beta = 2$.

Outline

For any graph $G = (V, E)$, let $\nu(G)$ denotes the maximum size of a matching and $\alpha(G)$ denotes the number of odd components of G. $U \subseteq V$ the graph obtained by deleting all vertices in U and edges incident with U,is denoted by $G \backslash U$.

Theorem (Tutte-Berge Formula)

Given a graph $G = (V, E)$*,*
$\nu(G) = min\{1/2(|V| - \alpha(G \backslash U) + |U|) : U \subseteq V\}$.

Theorem

For any graph $G = (V, E)$*, has a perfect matching if and only if* $\forall U \in V ; \alpha(G \backslash U) \leq |U|$.