MA1302 Engineering Mathematics I

Dr. G.H.J. Lanel

Integration

Outline

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- A function F is called an antiderivative of f if $\frac{d(F(x))}{dx} = f(x)$
- F is also known as an indefinite integral of f and denoted by

$$F(x) = \int f(x)dx$$

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- If F is an antiderivative of f then the most general antiderivative of f if $F(x) + c$ where c is an arbitrary constant.

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Common antiderivatives as follows;

f(x)	**F(x)**
$cf(x)$	$cF(x)(F' = f)$
$f(x) + g(x)$	$F(x) + G(x)(F' = f, G' = g)$
x^n	$\dfrac{x^{n+1}}{n+1} + c, (n \neq -1)$
$\dfrac{1}{x}$	$\ln \lvert x \rvert + c$
e^x	$e^x + c$

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f(x)	**F(x)**
$\sin(x)$	$-\cos(x)+c$
$\cos(x)$	$\sin(x)+c$
$\sec^2(x)$	$\tan(x)+c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)+c$
$\dfrac{1}{1+x^2}$	$\tan^{-1}(x)+c$

Example 01

Find the most general antiderivative of each of the following functions. (Check your answer by differentiation.)

E.g.01 $f(x) = 2$

Solution:

$$\int f(x)dx = \int 2dx$$

$$F(x) = 2x + c$$

c is an arbitrary constant
By differentiating,

$$\frac{dF(x)}{dx} = \frac{d(2x+c)}{dx}$$

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2. $f(x) = x^2 - x^{-2}$
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Example 02

Find f if, $f''(x) = 2x + 3$

Solution:

$$f''(x)dx = 2x + 3$$

$$f'(x) = \int (2x + 3)dx$$

$$f'(x) = x^2 + 3x + c$$

c is an arbitrary constant

$$f(x) = \int (x^2 + 3x + c)dx$$

$$f(x) = \frac{x^3}{3} + 3\frac{x^2}{2} + cx + d$$

, where d is an arbitrary constant.

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Exercise 02

Find f of the followings.

1. $f''(x) = x^3 - 2x^2 + 5$
2. $f'''(x) = \cos(x)$
3. $f'(x) = 1 + 3\sqrt{x}$, $f(4) = 25$
4. $f''(\theta) = \cos\theta + \sin\theta$, $f(0) = 3$, $f'(0) = 4$

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Outline

Definition

Let f be a continuous function defined on $a \leq x \leq b$ and let $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$ be endpoints of n sub-intervals of interval $[a, b]$. Let $\Delta x = \dfrac{b-a}{n}$ and $x_i^* \in [x_{i-1}, i]$ $(i = 1, 2, ..., n)$

Then, the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists. It gives the same value for all possible choices of sample points x_i^*. If it does exist, we say that f is integrable on $[a, b]$.

Note 01: The quantity

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is known as **Riemann sum.**

Note 02: The definite integral

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is a number and it does not depend on x.

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Theorem

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral

$$\int_a^b f(x)dx$$

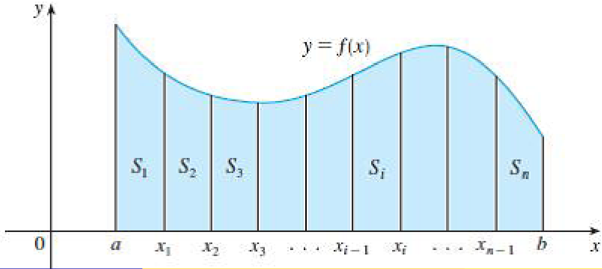
exists.

Outline

Exercise 05

Suppose we are given the function f shown here and we want to find the (shaded) area of the region bounded by the vertical lines $x = a$ and $x = b$, the x-axis and the graph of f.

1. Give an algebraic expression that approximates the shaded area.
2. Give an algebraic expression that is equal to the shaded area.



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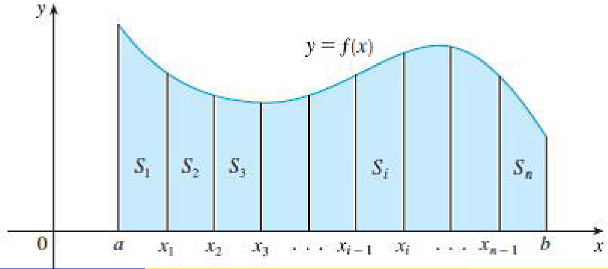

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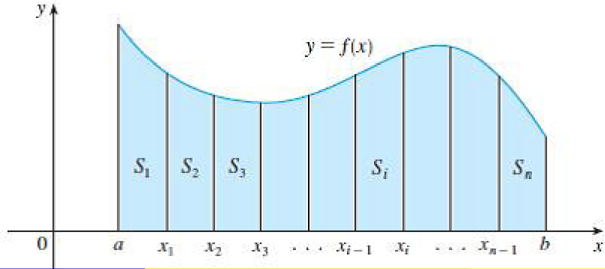
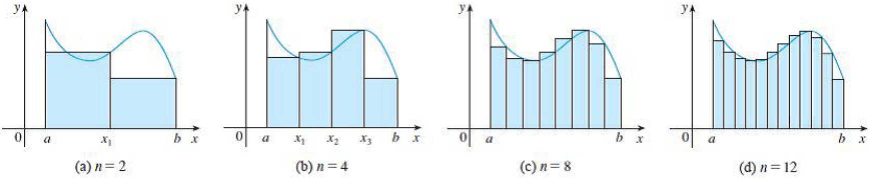


Note: We may choose x_i to be the left end point, right end point, mid point or any other point in the interval.

Approximated area for n = 2,4,8,12 with x_i chosen to be the right end point of each interval;



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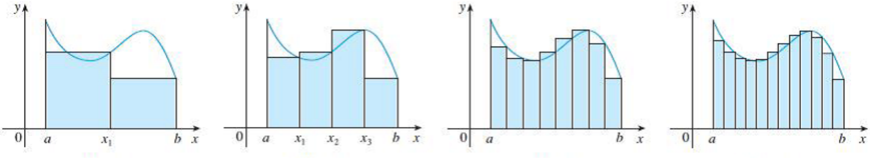

(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

Exercise 06

Suppose the odometer on a car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time(s)	0	5	10	15	20	25	30
Velocity(m/s)	17	21	24	29	32	31	28

1. Estimate the distance traveled during 30s.
2. Sketch a graph of velocity vs time and explain the relation among area under the graph, distance traveled, and the definite integral $\int_0^{30} v(t)dt$ where $v(t)$ denotes the velocity at time t.

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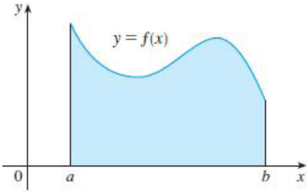
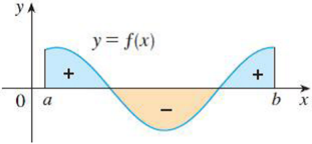
Note: If $f \geq 0$ for all $x \in [a, b]$ then the definite integral $\int_a^b f(x)dx$ represents the area under the curve $y = f(x)$, above the x-axis, from a to b.



Note: If f takes on both positive and negative values for $x \in [a, b]$ then the definite integral $\int_a^b f(x)dx$ represents the net area under the curve $y = f(x)$, above the x-axis, from a to b.



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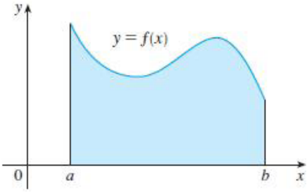

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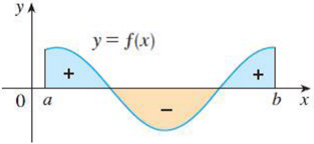
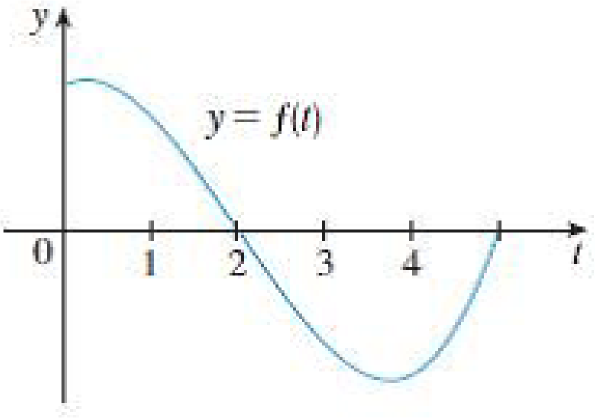


Exercise 07

Evaluate $\int_0^1 \sqrt{1 - x^2}dx$ by interpreting in terms of area.

Exercise 08

If $F(x) = \int_0^x f(t)dt$ where f is the function whose graph is given, estimate $F(i)$,$i = 0, 1, \cdots, 5$



Outline

- $$\int_a^a f(x)dx = 0$$
- $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$
- $$\int_a^b cdx = c(b-a)$$
- $$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

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- $$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

- $$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

- $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

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- $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

- $$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

- $$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

- $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

- If $f(x) \geq 0$ for $a < x < b$ then

$$\int_a^b f(x)dx \geq 0$$

- If $f(x) \geq g(x)$ for $a < x < b$ then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

- If $m \leq f(x) \leq M$ for $a < x < b$ then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

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Exercise 09

1. If $\int_1^5 f(x)dx = 12$ and $\int_4^5 f(x)dx = 5$ find $\int_1^4 f(x)dx$.
2. Find $\int_0^5 f(x)dx$ if
$$f(x) = \begin{cases} 3 & \text{if } x < 3 \\ x & \text{if } x \geq 3 \end{cases}$$
3. Show that $2 \leq \int_{-1}^1 \sqrt{1+x^2}dx \leq 2\sqrt{2}$

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Outline

1. Antiderivatives
2. Definite Integral
3. Definite Integral and Area
4. Properties of the Definite Integral
5. Evaluating Definite Integrals (Approximating by Numerical Methods)
6. The Fundamental Theorem of Calculus

Suppose that we divide the interval $[a, b]$ into n sub-intervals of equal length $\Delta x = \dfrac{b-a}{n}$, and x_i^* is any point in the ith sub-interval $[x_{i-1}, x_i]$. Then we have

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

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$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

1. Left endpoint approximating:

The point x_i^* is chosen to be the left endpoint of the interval. Then,

$$\int_a^b f(x)dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

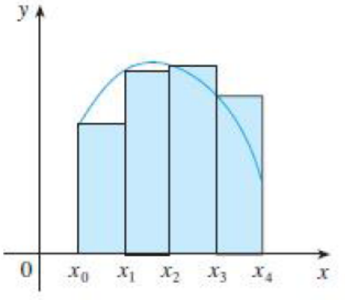

Figure: (a)Left endpoint approximation

2. Right endpoint approximating:

The point x_i^* is chosen to be the right endpoint of the interval. Then,

$$\int_a^b f(x)dx \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

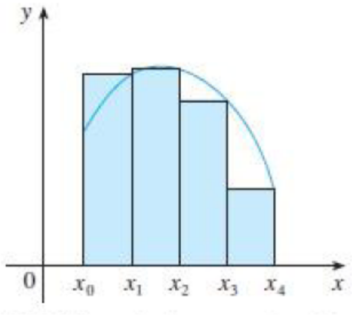

Figure: (b)Right endpoint approximation

3. Midpoint rule:

The point x_i^* is chosen to be the mid point $\bar{x}_i$ of the interval. Then,

$$\int_a^b f(x)dx \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$

Error bound: Error E_M of mid point rule is bounded by

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

where $|f''(x)| \leq K$ for $a \leq x \leq b$.

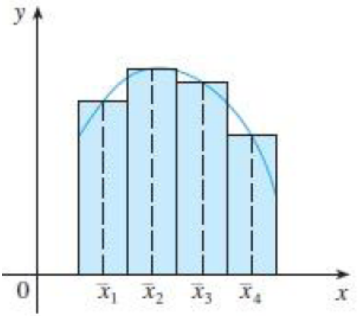

Figure: (c) Midpoint approximation

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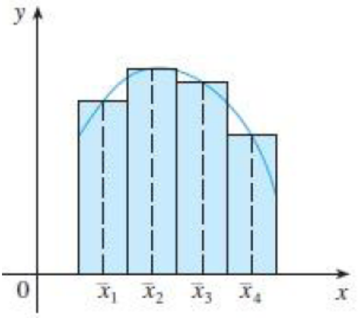

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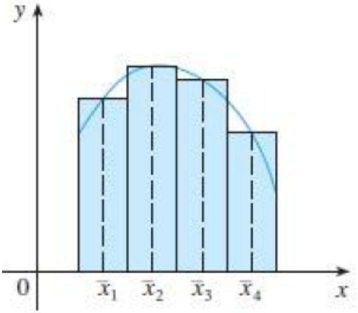

Figure: (c) Midpoint approximation

4. Trapezoidal rule:

Averaging left and right end approximations,

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Error bound: Error E_T of trapezoidal rule is bounded by

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

where $|f''(x)| \leq K$ for $a \leq x \leq b$.

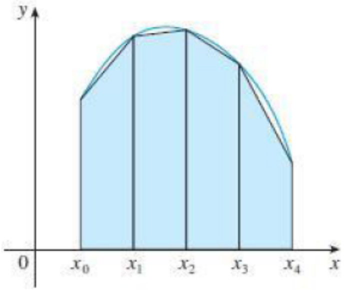

Figure: (d) Trapezoidal Approximation

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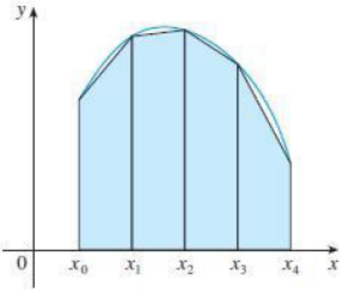

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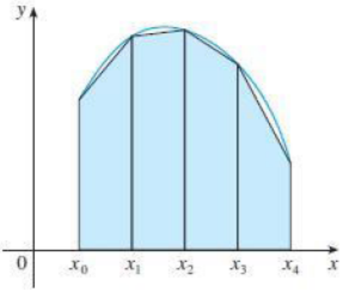


Figure: (d) Trapezoidal Approximation

5. Simpson's rule:

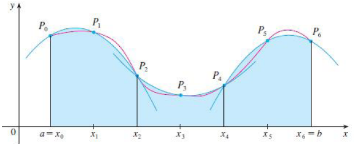
Let n be even. Using parabolas to approximate the curve (instead of lines),

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Error bound: Error E_S of Simpson's rule is bounded by

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

where $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$.



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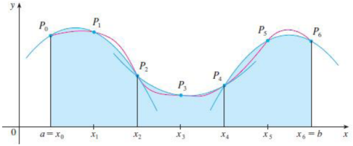

Exercise 10

1. Find the exact value of the definite integral $\int_1^2 \frac{1}{x} dx$. (Give your answer to 6 decimal places.)
2. Find the approximated value of the definite integral $\int_1^2 \frac{1}{x} dx$ using each method given above. (Give your answer to 6 decimal places.)
   1. Use left end approximation with n = 5,10 and 20.
   2. Use right end approximation with n = 5,10 and 20.
   3. Use mid point approximation with n = 5,10 and 20.
   4. Use trapezoidal rule with n = 5,10 and 20.
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Exercise 10 ctd..

Note: Error = [exact value of $\int_1^2 \frac{1}{x} dx$]-[approximated value].

- Calculate the error for each of above calculation.
- Calculate error bounds for mid point rule, trapezoidal rule, and Simpson's rule.
- Comment on your answers.

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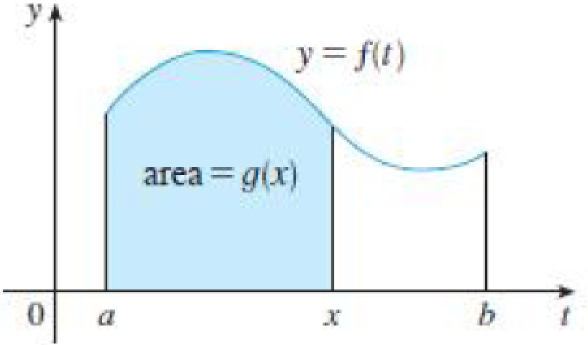
Exercise 11

How large should we take n in order to guarantee that the approximations for the definite integral $\int_1^2 \frac{1}{x} dx$ are accurate to within 0.0001 if we use; (a) mid point rule (b) trapezoidal rule (c) Simpson's rule?

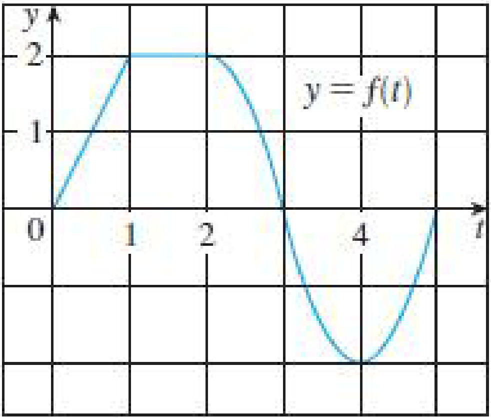
Outline

- Area function

Let $g(x) = \int_a^x f(t)dt$ where f is continuous and $a \leq x \leq b$. The function g is known as the area function; area under the graph of f from a to x, where x can vary from a to b.



If f is the function whose graph is shown and $g(x) = \int_0^x f(t)dt$ find $g(0), g(1), g(2), g(3), g(4), g(5)$ and sketch a rough graph of g.



Theorem

The Fundamental Theorem of Calculus (FTC)

Suppose f is continuous on $[a, b]$. Then,

Part 1. *If $g(x) = \int_a^x f(t)dt$ then $g'(x) = f(x)$. Alternatively*
$\frac{d}{dx} \int_a^x f(t)dt = f(x)$

Part 2. *If $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f, i.e., $F' = f$. Alternatively $\frac{d}{dx} \int_a^x f(t)dt = f(x)$*

Alternatively $\int_a^b F'(x)dx = F(b) - F(a) = F(x)|_a^b$

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Exercise 13

Apply the FTC 1 to find the derivative of each function.

1. $$g(x) = \int_1^x \frac{1}{1+u^3} du$$

2. $$g(x) = \int_x^0 \sqrt{1+t^2} dt$$

3. $$g(x) = \int_0^{x^2} \cos^3 \theta d\theta$$

4. $$g(x) = \int_1^{x^4} \sec(t) dt$$

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Exercise 14

Apply the FTC 2 to find the derivative of each function.

1. $$g(x) = \int_{-1}^{1} x^3 dx$$

2. $$g(x) = \int_{0}^{4} (4 - x)\sqrt{x} dx$$

3. $$g(x) = \int_{1}^{2} \frac{1 + u^4}{u^2} du$$

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Exercise 15

True or false?

$$\int_{-1}^{3} \frac{1}{x^2} = -\frac{4}{3}$$

Justify your answer.

Exercise 16 (Use any computational software.)

The sine integral function is defined as

$$Si(x) = \int_0^x \frac{\sin(t)}{t} dt$$

[The integrand $f(t) = \sin t/t$ is not defined when $t = 0$, but we know that it's limit is 1 when $t \longrightarrow 0$. So we define $f(0) = 1$ and this makes f a continuous function everywhere.]

1. Draw the graph of Si
2. At what values does this function have local maximum values?
3. Find the coordinates of the first inflection point to the right of the origin.
4. Does this function have horizontal asymptotes?
5. Solve the following equation correct to one decimal place: $\int_0^x \frac{\sin(t)}{t} dt = 1$