On the domination number of Hamiltonian graphs with minimum degree six

(Hua-Ming Xing, Johannes H. Hattingh, and Andrew R. Plummer, 2007)

Presented by Dr. G.H.J. Lanel

April 28, 2016

Outline

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Dominating set and domination number

Definition

- Throughout let $G = (V, E)$ be a simple graph with vertex set V and edge set E.
- A set $S \subseteq V$ is a **dominating set** if every vertex not in S has a neighbor in S.
- The **domination number** $\gamma(G)$, is the minimum size of a dominating set in G.

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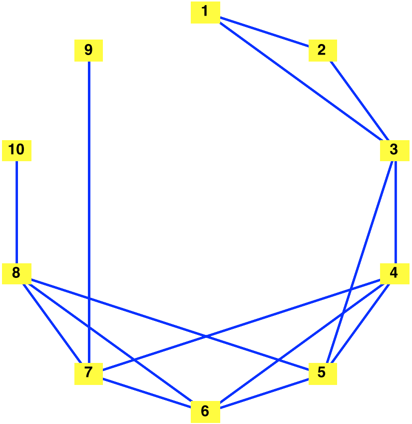

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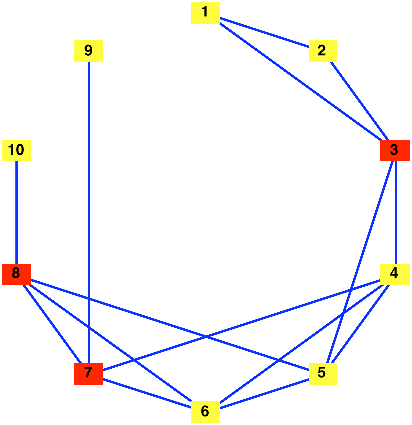

$\{3, 7, 8\}$ is a minimum dominating set (i.e., $\gamma(G) = 3$).

Order and minimum degree

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History

- O. Ore (1962) showed that if G is a graph of order n with $\delta(G) \geq 1$, then $\gamma(G) \leq \frac{n}{2}$.

- W. McCuaig and B. Shepherd (1989) showed that if G is a connected graph with $\delta(G) \geq 2$ and not one of the seven exceptional graphs, then $\gamma(G) \leq \frac{2n}{5}$.

- B. Reed (1996) showed that if $\delta(G) \geq 3$, then $\gamma(G) \leq \frac{3n}{8}$.

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- H. Xing, L. Sun, and X. Chen (2006) proved that Conjecture holds for graphs with $\delta(G) = 5$.
- The conjecture is open only for the graphs with $\delta(G) = 6$.
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Hamiltonian cycle and Hamiltonian graph

- A cycle containing every vertex of the graph is called a **Hamiltonian cycle**.
- A **Hamiltonian graph** is a graph possessing a Hamiltonian cycle.
- A **chord** of a cycle C is an edge not in C whose endpoints lie in C.

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A cowboy with a lasso

Lasso of a graph

- Let C be a cycle and P be a path of G with $V(C) \cap V(P) = \emptyset$.
- Let $v \in V(C)$ and x end vertex of P. Let $V' = V(C) \cup V(P)$ and $E' = E(C) \cup E(P) \cup \{vx\}$.
- The graph $L = (V', E')$ is called a **lasso** and the cycle C is called the **body** of L.

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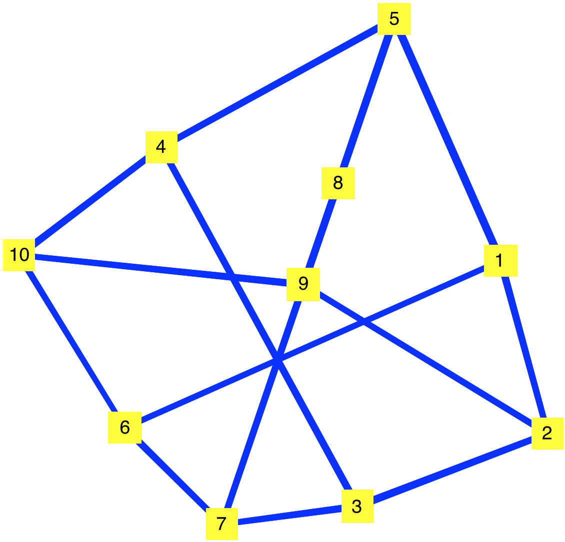

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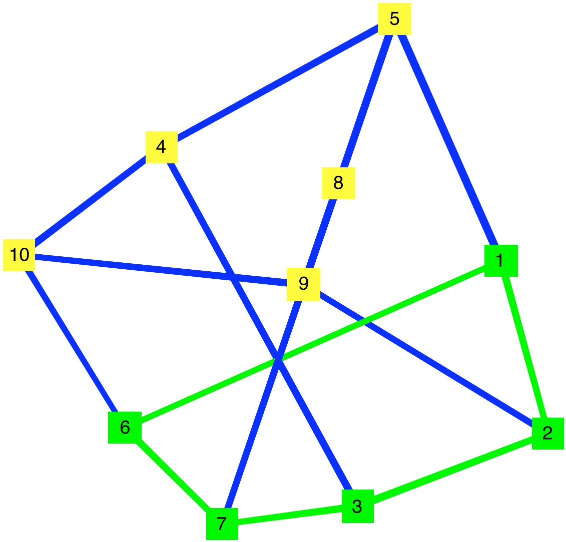

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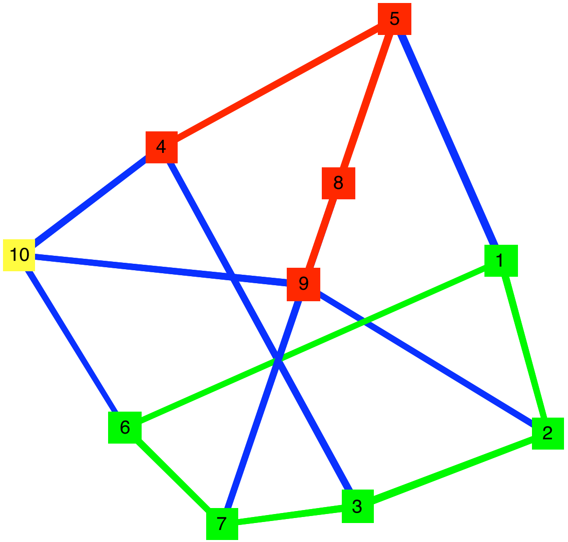

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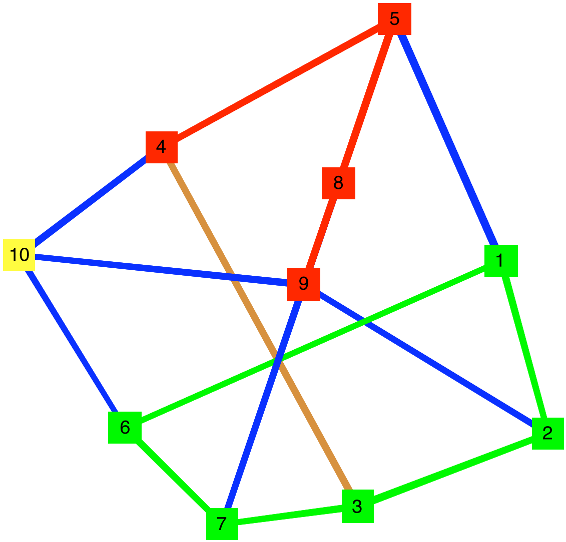

Lemma (⋆)

For $k \geq 1$, let $P = x_1, x_2, \ldots, x_{3k+1}$ be a path of order $3k+1$. If x_1 is adjacent to a vertex x_{3i} for some $1 \leq i \leq k$, then P can be dominated by k vertices.

Proof

The set $D = \{x_3, x_6, \ldots, x_{3k}\}$ is a dominating set of P such that $|D| = k$.

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Lemma ($\star\star$)

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Proof

Let $C = y_1, y_2, \ldots, y_{3k+1}$ and WLOG assume x_2 is adjacent to y_1. Then $D = \{x_2, y_3, y_6, \ldots, y_{3k}\}$ is a dominating set of $C \cup P$ such that $|D| = k+1$.

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Lemma (W.E. Clark and L.A. Dunning, 1997)

Let G be a graph of order n with $\delta(G) \geq 4$. If $n \leq 16$, then $\gamma(G) \leq \frac{n}{3}$.

Lemma (Xing, Sun, and Chen, 2006)

Let G be a graph of order $3m+1$, where $2 \leq m \leq 8$. If $\delta(G) \geq 5$, then $\gamma(G) \leq m$.

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Theorem

Let G be a Hamiltonian graph of order n such that $\delta(G) \geq 6$*. Then* $\gamma(G) \leq \frac{6n}{17}$.

Proof

- *Let* $V(G) = \{1, 2, \ldots, n\}$ *and WLOG assume* $C = 1, 2, \ldots, n, 1$ *is a Hamiltonian cycle of G.*
- *If* $n \leq 16$*, then by Lemma (Clark and Dunning),* $\gamma(G) \leq \frac{n}{3} \leq \frac{6n}{17}$*. Thus,* $n \geq 17$*.*
- *Let* $m \geq 6$*, then there are 3 cases to be considered.*

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Case

$n = 3m - 1.$

- The $D = \{2, 5, \ldots, 3m-1\}$ is a dominating set of G such that $|D| = m = \frac{n+1}{3}$.
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- If $m \le 8$, then $n \le 25$, and by Lemma (Xing, Sun, and Chen), $\gamma(G) \le m \le \frac{6n}{17}$.
- If $m \ge 11$, then $n \ge 34$, and $D = \{2, 5, \ldots, 3m+1\}$ is a dominating set of G such that $|D| = m + 1 = \frac{n+2}{3}$. It follows that $\gamma(G) \le \frac{n+2}{3} \le \frac{6n}{17}$.
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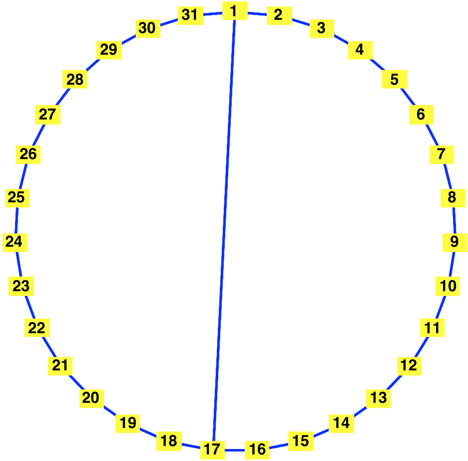

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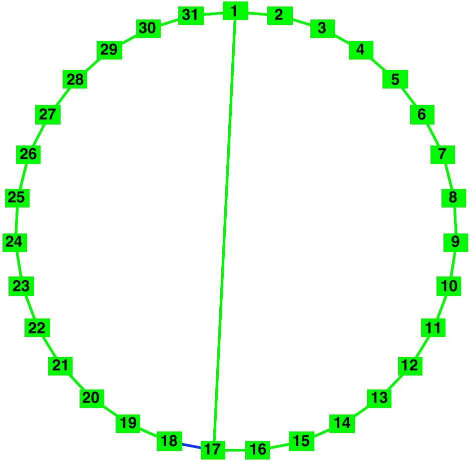

- Let $v \in V(G)$. WLOG assume that $1v$ is a chord of C and $1, v, v-1, \ldots, 1$ is the body of L.

- Consider possible values for v. By Lemma($\star$), we may assume that 1 is not adjacent to $3i$ for all i. Similarly 31 is not adjacent to $3i-1$ for all i.

- Since the body of L is maximum, and by relabeling if necessary, we have that $v \geq 17$. So, $v \in \{17, 19, 20, 22, 23, 25, 26, 28, 29\}$.

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- We know that 31 is adjacent to 1 and 30. Similarly 30 is adjacent to 31 and 29.
- Suppose b is adjacent to 31. Then we obtain lassos L_1 and L_2 with cycle lengths $b + 1$ and $32 - b$.
- Thus $b + 1 \leq v$ and $32 - b \leq v \Rightarrow 32 - v \leq b \leq v - 1$.
- Similarly if c is adjacent to 30, then $31 - v \leq c \leq v - 2$.

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$v = 17.$

- Consider the vertex 31. We have that 31 is adjacent to vertices 1 and 30.
- Since $32 - 17 \leq b \leq 17 - 1 \Rightarrow 15 \leq b \leq 16$, 31 is possibly adjacent to vertices in $\{15, 16, 1, 30\}$, a contradicting fact that $deg(v) \geq 6$.

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$v \in \{19, 22, 25, 28\}.$

A contradiction by Lemma($\star\star$).

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A contradiction by Lemma($\star\star$).

Case

$v = 17$.

- Consider the vertex 31. We have that 31 is adjacent to vertices 1 and 30.
- Since $32 - 17 \leq b \leq 17 - 1 \Rightarrow 15 \leq b \leq 16$, 31 is possibly adjacent to vertices in $\{15, 16, 1, 30\}$, a contradicting fact that $deg(v) \geq 6$.

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A contradiction by Lemma($\star\star$).

Case

$v = 20.$

- Consider the vertex 31. We have that 31 is adjacent to 1, 30 and possibly 12, 13, 15, 16, 18, 19.
- Since deg(31) ≥ 6, 31 must be adjacent to at least one of the vertices 12, 15, or 18.

Case

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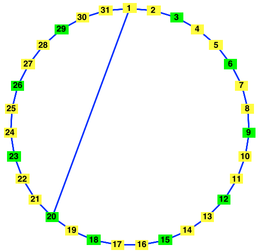

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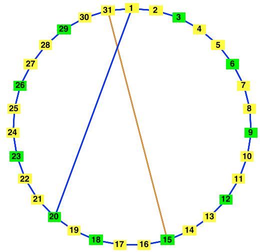

Then $D = \{3, 6, 9, 12, 15, 18, 20, 23, 26, 29\}$ is a dominating set with $|D| = 10$, a contradiction.

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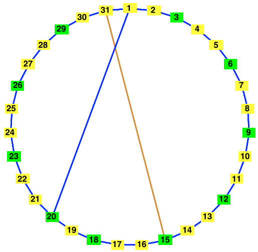

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Case

$v = 23.$

Similar to the case $v = 20$.

Case

$v = 26.$

Similar to the case $v = 20$.

Case

$v = 29.$

Similar to the case $v = 20$.

Thank You!