Complex Root Isolation

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Outline

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- Let A be a squarefree univariate polynomial with Gaussian integer coefficients and positive degree.
- Isolating all the zeros of A into disjoint rectangles is called complex root isolation.

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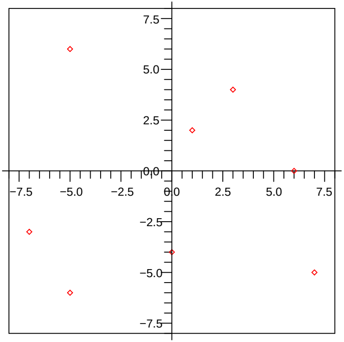
Example

- Let $A(z) = (z - 1 - 2i)(z - 3 - 4i)(z - 6)(z + 7 + 3i)(z + 4i)(z + 5 - 6i)(z + 5 + 6i)(z - 7 + 5i)$.



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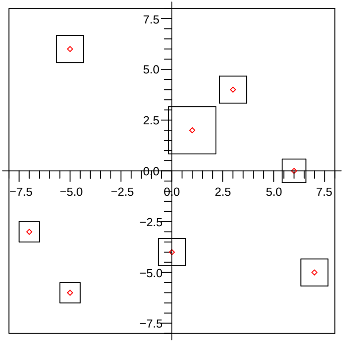

Reduction

- The root bound of a polynomial A defines an initial square which contains all the zeros of A.
- The initial square is then bisected, and the number of zeros in each half is determined.
- A half which contains exactly one zero is appended to the solution list and a half which contains more than one zero is appended to the working list.
- The bisection process continues for rectangles in the working list.
- Hence the complex root isolation of A can be done by reducing the problem to determining the number of zeros of A in any given rectangle.

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Number of Zeros in a Rectangle

- Let $A(z)$ be a squarefree polynomial and R a rectangle in the complex plane.
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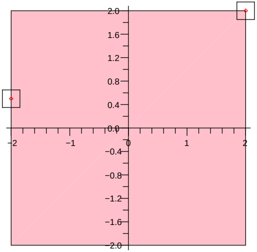
- Let $A(z) = (z - a_1)(z - a_2)\cdots(z - a_n)$ and $R := [-2, 2] \times [-2, 2]$. Suppose there are two zeros of A on ∂R.



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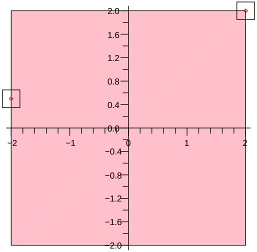

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History

- There are three algorithms by Pinkert, Wilf, and Krandick for complex root isolation of a polynomial A.
- In the Pinkert, if $A(z) \in \mathbb{Z}[z]$, then the complex zeros of A have never been isolated.
- In Wilf's algorithm, the subdividing process terminates when there are zeros on the boundary resulting in failure of the algorithm.
- In the algorithm of Collins and Krandick, the problem is circumvented without justification.
- In this thesis we rectify all these problems by computing exact number of zeros of A in a rectangle R when there is a zero of A on the boundary of R.

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Main Theorem

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Let $A(z)$ be a squarefree polynomial and R a rectangle in the complex plane. Then the number of zeros of A in the interior of R is given by

$$m = \frac{1}{2\pi}[\sum_{i=1}^{s} pac(z_i) + \sum_{j=1}^{t} pac(p_j)]$$

where $z_1, z_2, \ldots, z_s$ are the zeros of A and $p_1, p_2, \ldots, p_t$ are the critical points of A on ∂R.

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Pseudo argument change (*pac*)

Definition

- Let O be the origin and let w_1 and w_2 be non-zero points on the complex plane.
- Suppose n is the number of times that the ray $\overrightarrow{Ow}$ approaches or leaves an axis as it rotates from $\overrightarrow{Ow_1}$ to $\overrightarrow{Ow_2}$ counterclockwisely.
- Then the **pseudo argument change** from w_1 to w_2 is defined to be,

$$pac(w_1, w_2) = \begin{cases} \frac{\pi}{4} n & \text{if } \mathbf{Re}(w_1) * \mathbf{Re}(w_2) \leq 0 \text{ or } \mathbf{Im}(w_1) * \mathbf{Im}(w_2) \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

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Properties for a zero of A on ∂R

Let $A_1(x, y)$ and $A_2(x, y)$ be the **real** and **imaginary** parts of A. Suppose $z_0 = x_0 + iy_0$ is a zero of A on an edge E of R.

- If z_0 is not a corner of R, then there exist a, b on E as z_0 such that one of the following holds:

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    i. There are no zeros of A, A_1, or A_2 on $\overline{az_0}$ and $\overline{z_0b}$ except z_0.

    ii. One of the segments $\overline{az_0}$ or $\overline{z_0b}$ contains no zeros of A_1 or A_2 (except z_0) and the image of the other under A lies on a coordinate axis.

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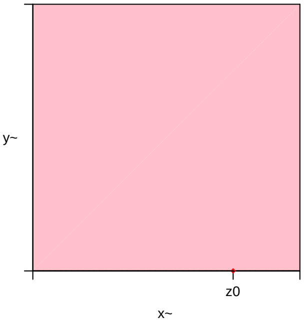
$pac(z_0)$

- Let a and b be two points satisfying the above properties such that a, z_0, and b are oriented counterclockwise on ∂R as follows.



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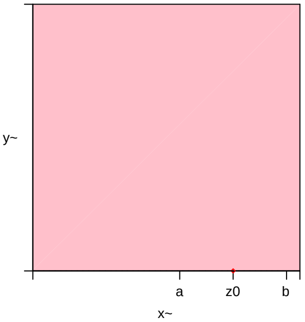

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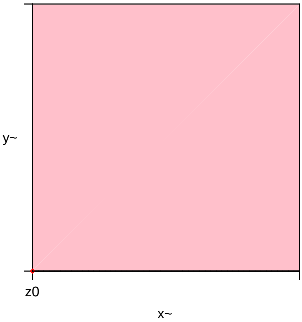

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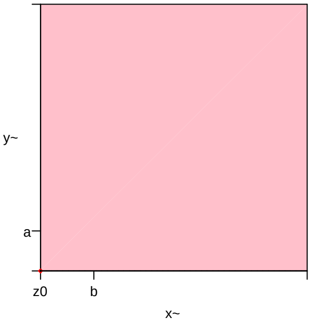

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- Note that $pac(z_0)$ is **independent** of choice of a and b.

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- If $a \in \partial R$ is not a zero of A and $A_1(a) = 0$ or $A_2(a) = 0$, then a is called an **axial point.**
- Suppose p is a point on an edge E of R, which is not a zero of A.
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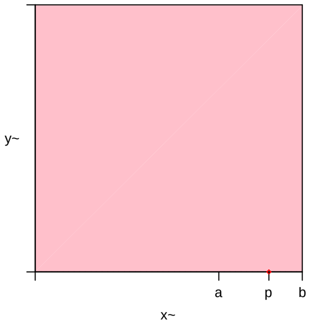
- Let a, b satisfy the above properties such that a, p, and b are oriented counterclockwisely on ∂R, as in the illustration.



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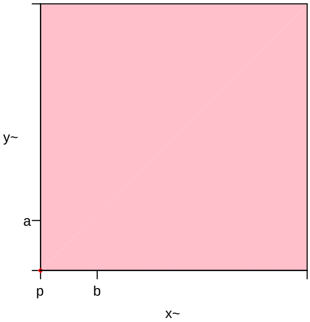

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Half-planes

- A **half-plane** associated with line d consists of all the points on one side of d.
    - If all the points on d are included, then it is **closed**.
    - If none of the points on d are included, then it is **open**.
- Let $z_0 = x_0 + iy_0$ be a zero of A.
    - A left (right) half-plane at z_0 is the half-plane left (right) of the line $z = x_0$.
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Quadrants

- The four regions create by intersection of two lines $z = x_0$ and $z = y_0$ at z_0 are called **quadrants** at z_0.
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Quadrants

- The four regions create by intersection of two lines $z = x_0$ and $z = y_0$ at z_0 are called **quadrants** at z_0.
- If all the points of the boundary are included, then it is **closed**.
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Lemma

If $z_0 = x_0 + iy_0$ is a zero of a squarefree polynomial $A(z) \in \mathbb{C}[z]$. Then (x_0, y_0) is a simple point on the curves $A_1(x, y) = 0$ and $A_2(x, y) = 0$.

Remark

Suppose $B = A_1$ or A_2. Then (x_0, y_0) is a simple point of B.

If $\frac{\partial B(x_0,y_0)}{\partial y} \neq 0$, then by the Implicit Function Theorem, there exists an $h > 0$ and a unique, real-valued function $\varphi(x)$ defined for $|x - x_0| < h$ such that $\varphi(x_0) = y_0$ and $B(x, \varphi(x)) = 0$.

Similar results can be stated when $\frac{\partial B(x_0,y_0)}{\partial x} \neq 0$.

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Theorem

Let $z_0 = x_0 + iy_0$ be a zero of a squarefree polynomial $A(z) \in \mathbb{C}[z]$. Let A_1 and A_2 be real and imaginary parts of A. Then there exists a rectangle R_0 around z_0 in the complex plane such that the following holds:

i. *The number of critical points on ∂R_0 in a closed half-plane at z_0 is 1, 2, or 3.*
ii. *The number of critical points on ∂R_0 in a closed quadrant at z_0 is 0, 1, or 2.*
iii. *There are exactly 4 critical points on ∂R_0.*

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Outline of the proof

- Since (x_0, y_0) is a simple point on A_1 (A_2), using Cauchy-Riemann equations, either
- $\frac{\partial A_1(x_0,y_0)}{\partial x} \neq 0$ and $\frac{\partial A_1(x_0,y_0)}{\partial y} \neq 0$ or
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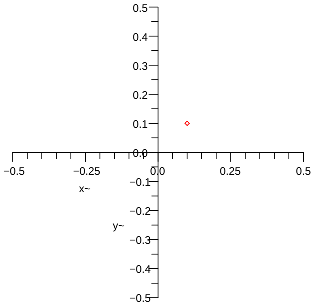

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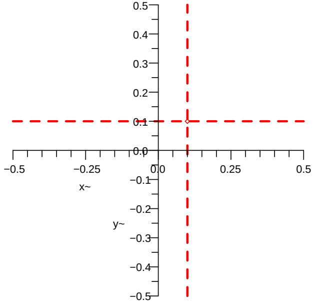

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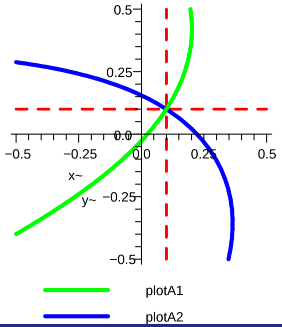

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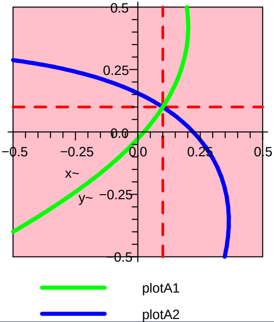

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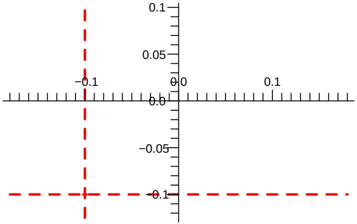

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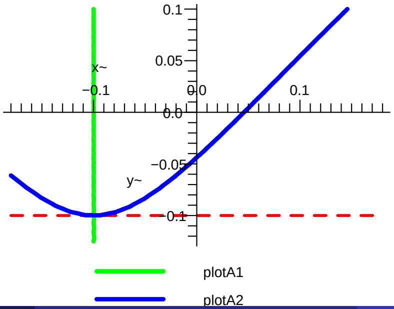

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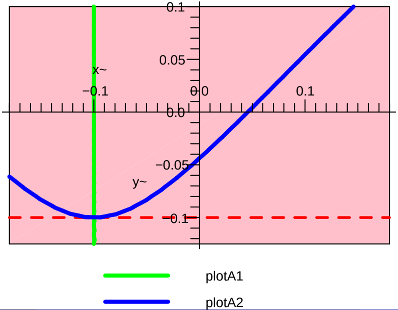

Corollary

Let R_0 be the rectangle around z_0 given by Theorem. As z traverses ∂R_0 counterclockwise, $A(z)$ always approaches and leaves the axes in counterclockwise direction.

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On the $pac(z_0)$

- If a and b are the intersection points of ∂R and ∂R_0 such that a, z_0, and b are oriented counterclockwise on ∂R, then $pac(z_0) = -pac(A(b), A(a))$.

- To compute $pac(z_0)$ one only needs to compute $pac(A(b), A(a))$ which by the above Theorem, is dependent only on the signs of the real and imaginary parts of $A(b)$ and $A(a)$.

- If z_0 is not a corner point of R, then the possible values for $pac(z_0)$ are $-\pi/2$, $-\pi$, and $-3\pi/2$.

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Proof of the main Theorem

The proof follows from the following Lemma and the argument principle.

Lemma

Let R_i be the rectangle around z_i given by the above Theorem and let $\overline{R} = R - \cup_{i=1}^{s} R_i$. If $C = \partial \overline{R}$, then

$$\Delta_C argA(z) = \sum_{i=1}^{s} pac(z_i) + \sum_{j=1}^{t} pac(p_j),$$

where $\Delta_C argA(z)$ denotes the net change in $argA(z)$ as the point z traverses C once over in the counterclockwise direction.

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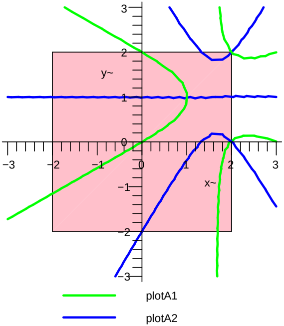
Example: $A = (z-2)(z-1-i)(z-2-2i)$



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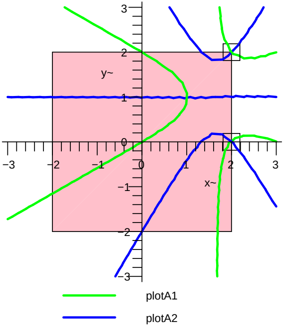

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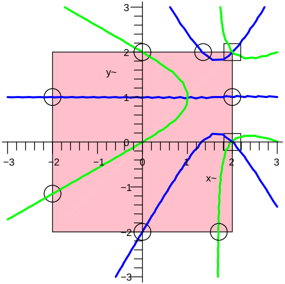

$m = \frac{1}{2\pi}[\sum_{i=1}^{2} pac(z_i) + \sum_{j=1}^{7} pac(p_j)] = \frac{1}{2\pi}(-\pi - \pi/2 + 7\pi/2) = 1.$

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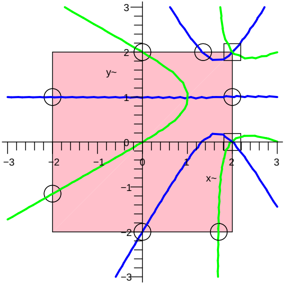

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Outline

Algorithm: ComplexRootsIsolation(A)

Input: $A(z)$, squarefree, with positive degree and Gaussian integer coefficients.

Output: L, a list of isolating rectangles for all zeros of $A(z)$.

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Theorem

Let A be a polynomial with Gaussian integer coefficients and degree n and let $d = \|A\|_1$.
Then the complexity of the algorithm **ComplexRootsIsolation**(A) *is dominated by $O(n^7 L(n)^3 L(nd))$.*

1. The following table reports the computing time of polynomials with coefficient bit-length 32 and the probability p that a polynomial has a non-zero coefficient.

Degree	Time (second)			
	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 1$
10	0.044	0.032	0.048	0.092
20	0.268	0.332	0.539	0.528
30	0.616	1.324	2.328	2.616
40	2.284	3.604	5.004	6.668
50	4.848	6.252	11.484	22.385
60	12.716	15.684	25.305	44.738
70	28.021	28.637	52.491	56.551
80	36.374	57.639	89.025	96.166
90	53.535	91.741	155.705	177.019
100	82.609	164.494	190.891	217.257

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Table : CPU time for polynomials with coefficient bit-length 32

2. The following table shows the computing time of polynomials with degree 50 and the probability p that a polynomial has a non-zero coefficient.

Coefficient bit-length	Time (second)			
	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 1$
5	4.884	6.308	11.419	22.357
10	5.164	6.748	12.364	24.229
15	5.539	7.159	13.224	26.237
20	6.984	9.428	13.359	21.389
25	7.448	9.892	14.019	22.653
30	6.508	13.488	15.012	20.713
35	6.939	11.648	14.492	22.197
40	7.296	12.188	15.456	23.161
45	9.164	19.585	26.085	27.761
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Observation

The results shows that the average computing time of the algorithm in practice seems to be approximately

- **cubic** in the degree of the input polynomial and
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Outline

Future works

- It might be interesting to see whether the theoretical complexity of our algorithm can be improved.
- The algorithm by Wilf uses Sturm sequences and Cauchy indices to determine the number of zeros in a rectangle. It works with a rectangle and its four sub-rectangles and assume there are no zeros on the boundary.

It might be interesting to see if Wilf algorithm can be modified to produce rectangles which contain exactly one zero by using our mathematical results.

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Thank You!